BASIC CONTROL THEORY

KINEMATICS
DYNAMICS
CONTROL

Qualitative differences between kinematics, dynamics and control.

Kinematics is just geometry, in our case in 3D. There is therefore a certain objectivity to its foundations.

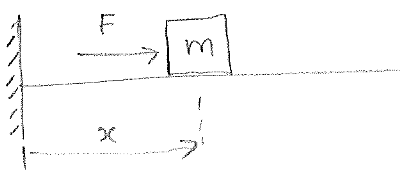
Dynamics is part of physics and therefore also enjoys a certain objectivity. Two people writing equations for the dynamics of a system should get the same dynamical behavior

In robotics, Control theory builds on kinematics and dynamics. It is the study of how to make objects behave in a desired manner in various situations. While control theory does enjoy a reasonably sound mathematical foundation, there is an enormous diversity of control schemes and control techniques which produce dynamical behavior of one or another kind. Some of these techniques can be ad hoc, and some provably good. Often, there are multiple correct answers to the same question, etc. In this sense, control theory seems a bit less objective than kinematics or dynamics.

SIMPLEST CONTROL SCHEMES FOR "POSITION CONTROL". ("POSITION REGULATION")

(LINEAR CONTROL) (Chapter 9 from TEXTBOOK).

Consider the following simple situation. A mass sitting on a frictionless table, with an external "control force" F that is available to control the position of the mass m.



Equation of motion.

$$m\ddot{x} = F$$

Say our goal is to position the mass at $x = x_{desired}$.

Strategy 1 PROPORTIONAL CONTROL : If the $x \neq x_{desired}$, then use a "restoring force" toward $x_{desired}$, and proportional to how far x is from $x_{desired}$.

i.e., $F = -k_p(x - x_{desired})$

So $m\ddot{x} = F$
$= -k_p(x - x_{desired})$

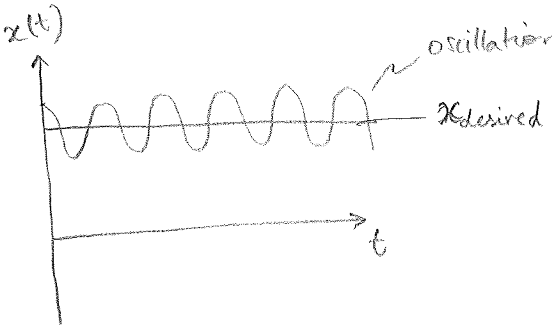
$$\Rightarrow \boxed{m\ddot{x} + k_p x = k_p x_{desired}}$$

$$m\ddot{x} + \underbrace{k_p x}_{\text{Spring-like term}} = \underbrace{k_p x_{desired}}_{\text{constant term}}$$

Ok, how does this system behave?
Observe that this equation is identical to a spring-mass system with no damping.

If x starts out from $x_{desired}$ $\left[x(0) = x_{desired}\right]$,
$x(t) = x_{desired}$ for all time.

If $x(0) \neq x_{desired}$, $x(t)$ is oscillatory for ever, because there is no damping.



Oscillation forever is not desired behavior. We would like the oscillations to die down. So we add a damping-like term as below.

PROPORTIONAL - DERIVATIVE CONTROL (PD control)

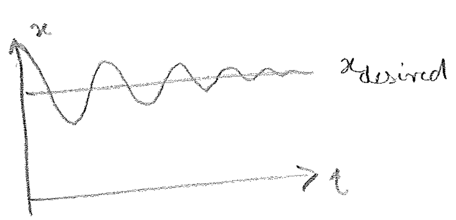
Now we $F = \underbrace{-k_p(x - x_{desired})}_{\text{proportional term}} \underbrace{- k_v \dot{x}}_{\text{derivative term}}$] provides damping

So we have

$$m\ddot{x} = -k_p(x - x_{desired}) - k_v\dot{x}$$

$$\boxed{m\ddot{x} + \underbrace{k_v\dot{x}}_{\text{damping-like}} + \underbrace{k_p x}_{\text{spring-like}} = k_p x_{desired}}$$

← identical to a mass-spring-damper system.



So all oscillations die down to $x_{desired}$, as $t \to \infty$.

How do we know that $x \to x_{desired}$? We compute the "steady state" of the system by setting all the time-derivative terms to zero.

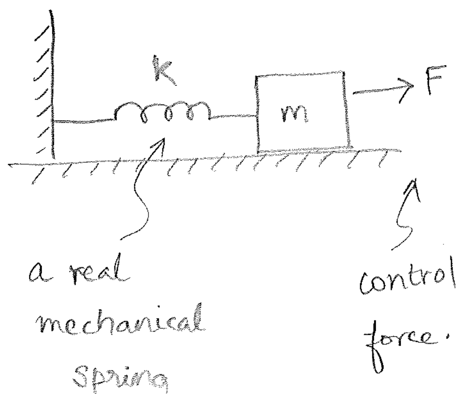
$$m\cancelto{0}{\ddot{x}} + k_v \cancelto{0}{\dot{x}} + k_p x = k_p x_{desired}$$

So $x_{steady} = x_{desired}$.

So that's nice. All oscillations die down (due to damping) and they die down to $x_{desired}$.

BUT PD control does not always converge to $x_{desired}$.

We only need to consider a slightly more complex system. One with an intrinsic stiffness..



Now, the equation of motion is

$$m\ddot{x} + kx = F$$

$$F = -k_p(x - x_{desired}) - k_v \dot{x}$$

So,

$$m\ddot{x} + kx = -k_p(x - x_{desired}) - k_v\dot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + (k + k_p)x + k_v\dot{x} = k_p x_{desired}}$$

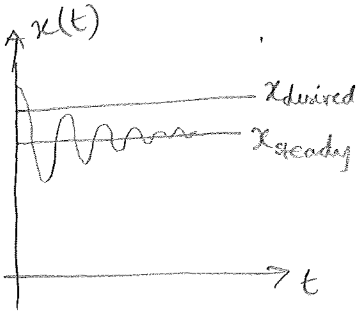
constant term.

also a mass-spring-damper system

What is the steady state now?

Set $\ddot{x} = 0$ $\dot{x} = 0$, we have

$$0 + (k + k_p)x + 0 = k_p x_{desired}.$$



$$\boxed{x_{steady} = \frac{k_p}{k + k_p} x_{desired}} \neq x_{desired}.$$

"STEADY STATE ERROR"

That is, the steady state is a different from $x_{desired}$!

How can we make the steady state closer to $x_{desired}$?

$$x_{steady} = \frac{k_p}{k+k_p} x_{desired} = \frac{1}{k/k_p + 1} \cdot x_{desired}.$$

when $k_p >> k$, $x_{steady} \rightarrow x_{desired}$ ["asymptotic convergence"]

That is, making the proportional gain k_p very large makes the steady state error go away.

BUT, making k_p large has its disadvantages:

→ it makes the system "very stiff"

→ small deviations from $x_{desired}$ produce high control forces

→ high frequency oscillations / fast transients (which might be bad sometimes).

Ok, can we get asymptotic convergence to $x_{desired}$ without having to increase k_p ?

YES, add an integral term, as below.

PROPORTIONAL - INTEGRAL - DERIVATIVE CONTROL

$$F = -k_p(x - x_{desired}) - k_v\dot{x} - k_i\left(\int_0^t (x - x_{desired})\, dt\right)$$

So we have

$$m\ddot{x} + kx = -k_p\left(x - x_{desired}\right) - k_v\dot{x} - k_i\left[\int_0^t \left(x - x_{desired}\right) dt\right].$$

$$\boxed{m\ddot{x} + \left(k+k_p\right)x + k_v\dot{x} + k_i\int_0^t \left(x - x_{desired}\right) dt = k_p x_{desired}.} \quad - \circledast$$

What are the steady states of this "integro-differential" equation?
First convert it into a differential equation. That is, remove the integral term.

Introduce variable $y = \int_0^t \left(x - x_{desired}\right) dt.$

(or) equivalently $\dot{y} = x - x_{desired}.$

So the equation ($\circledast$) becomes the following 2 differential equations:

$$\boxed{\begin{aligned} & m\ddot{x} + \left(k+k_p\right)x + k_v\dot{x} + k_i y = k_p x_{desired} \\ & \dot{y} = x - x_{desired}. \end{aligned}}$$

What is the steady state of these differential equations?
Set all time derivative terms to zero.

$\ddot{x} = 0 \quad \dot{x} = 0$ and $\dot{y} = 0$

$$\Rightarrow \quad 0 + \left(k+k_p\right)x_{steady} + 0 + k_i y_{steady} = k_p x_{desired}$$

$$0 = x_{steady} - x_{desired}.$$

$\Rightarrow$ $\boxed{x_{steady} = x_{desired}.}$ ← This is what we wanted.

and $y_{steady} = -\dfrac{k}{k_i} x_{steady}$. ← We don't really care about this.

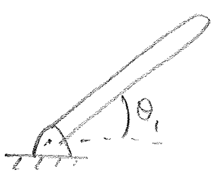
So no steady state error when we have the integral term.

While the integral term does kill the steady state error, there are some disadvantages:

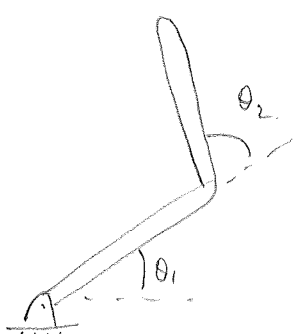
- if k_i is too large, the system can go unstable.
- if k_i is not too small, the convergence time-scale may get increased.

Some examples in MATLAB

Once you've written a program to simulate the DYNAMICS, its easy to add simple position control. Just a couple of more lines.

1) 

use $$\tau_1 = -k_p\left(\theta_1 - \theta_{1,desired}\right) - k_v \dot{\theta}_1 - k_i \int_0^t \left(\theta_1 - \theta_{1\,desired}\right) dt$$

2) 

Use $$\tau_1 = -k_p\left(\theta_1 - \theta_{1desired}\right) - k_v \dot{\theta}_1 - k_i \int_0^t \left(\theta_1 - \theta_{1desired}\right) dt.$$

and $$\tau_2 = -k_p\left(\theta_2 - \theta_{2desired}\right) - k_v \dot{\theta}_2 - k_i \int_0^t \left(\theta_2 - \theta_{2desired}\right) dt.$$