Homework 1, Nonlinear Dynamics, Spring 2016
ODEs in one independent variable, bifurcations, and other basic stuff

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Due: Friday, Jan 29, 2016, in class or by 7 pm

Total: 50 points. Some of the following questions will require you to recall basic undergraduate calculus and dynamics knowledge. Please read the course policy handout again and follow the guidelines therein.
Be sure to upload any MATLAB programs you write on Carmen. In a compressed folder. Name the folder "LastNameHW1" and then compress it. Please follow the naming guidelines to the letter. No need to print the programs, but do print it's output, for instance, graphs, etc. The HW submission should be a self-contained document. Thanks.

Q1. Stretched linear string as a nonlinear spring [18pts]. Consider the taut horizontal string (or rubber band) discussed in lecture-1. The rest (stress-free, relaxed) length of the string is $2l_s$, the initial horizontal length of the string is $2l_0$, the axial stiffness (spring constant) of the string is k, and the initial tension in the stretched string is $T_0 = k(2l_0 - 2l_s)$. Use notation as in lecture.

a) Derive the full nonlinear relationship between vertical force F at the midpoint of the string and the vertical displacement x at the midpoint. Plot $F(x)$ to see whether this system behaves like a hardening or softening spring (local stiffness/slope increases with $|x|$ or decreases with $|x|$). Show that the rate of change of slope (stiffness) $F''(x)$ is positive for $x > 0$. [5pts]

b) Derive a linear approximation to the nonlinear spring force $F(x)$ when x is very small ($x \ll l_0$). Show that the linear stiffness is proportional to the initial tension T_0 in the string. [4pts]

c) What happens to the above linear approximation when $l_0 = l_s$, or equivalently, when the initial tension $T_0 = 0$? [2pts]

d) When $l_0 = l_s$, obtain a low order (nonlinear) polynomial approximation to $F(x)$ of the form $F(x) = cx^\alpha$. [5pts]

e) Does anything interesting happen to $F(x)$ when $l_s = 0$? [2pts]

f) Write the equation of motion for a mass m at the center of the string, in the presence of downward gravitational force mg. Approximate the nonlinear ODE by replacing the nonlinear term(s) by its Taylor series about some point $x = x_0$, until the cubic term. Under what conditions do/don't we get a quadratic term? Consider $x_0 = 0$ and $x_0 \neq 0$.

Notes. You might use Taylor series to obtain linear or higher order polynomial approximations about $x = 0$. You might use the discussion in class for answering some of these questions. You might use ad hoc approximations like $\sqrt{x^2 + l_0^2} \approx l_0(1 + x^2/2l_0^2) \approx l_0$, when $x \ll l_0$. You might use `MATLAB`'s symbolic toolbox to obtain taylor series and derivatives (you should probably learn to use this anyway).

Notes. Even a simple physical situation such as a taut rubber band, modeled as a linear spring, can give rise to nonlinear stiffness when one considers large displacements. One can do similar derivations for other elastic structures (like beams, plates, etc) starting from the appropriate partial differential equation, assuming large deflections.

Q2. Growth rates: exponentials, logarithms and polynomials [8pts]. If you have two functions $f_1(t)$ and $f_2(t)$, one way to show that $f_1(t)$ is eventually ("asymptotically") larger than $f_2(t)$ as $t \to \infty$ is to show that $\lim_{t\to\infty} f_2(t)/f_1(t) = 0$. (This is a sufficient but not necessary condition).

a) Show that $x_0 e^{kt}$ for $x_0, k > 0$ is always eventually larger than any function of the form $y_0 t^n$, for any

$y_0, n > 0$, as $t \to \infty$. [3pts]

b) When you want to earn interest on some saved money and you know that you will live an arbitrarily long time (perhaps forever), show that "compound interest" ($y(t) = y_0(1+k_1)^{k_2 t}$) is always better by "simple interest" ($x(t) = x_0(1+k_3 t)$), whatever initial amount you start with and whatever the respective interest rates. [3pts]

c) Show that as $t \to \infty$, $f_1(t) = a_1 \log(t)$ is always overtaken by $f_2(t) = a_2 t^k$, for any $k, a_1, a_2 > 0$. [2pts]

Moral. Exponentials grow faster than polynomials and polynomials grow faster than logarithms.

Q3. Bead on a horizontal wire: Fixed points, Stability, and Bifurcations (adapted from Strogatz). [18pts] A bead of mass m is constrained to slide along a straight horizontal wire. A spring of rest (relaxed) length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire. See Figure. 1. Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

a) Write the equation of motion for the bead. (Consider $F = m\ddot{x}$ in the horizontal direction) [4pts].

b) Suppose $m = 0$, so that the equation of motion reduces to a first order equation in x. Find all possible equilibria, i.e., fixed points, x^* as functions of k, h, b, and L_0. Look out for cases $L_0 < h$ and $L_0 > h$ [4pts].

c) Using Linear stability analysis, classify the stability of all the fixed points x^*. Draw a bifurcation diagram with all the fixed points x^* on the vertical-axis and L_0 on the horizontal axis. Show the unstable branches of the fixed point using a dotted line. [6+1=7pts]

d) If the equation of motion from part-b was of the form $\dot{x} = f(x)$, sketch f as a function of x, possibly based on your answers from part-c. [3pts]

e) Solve both the first order and the second order ODE with `ode45` in `MATLAB` for the same $x(0)$. For fixed values of all other parameters, plot the second order ODE solution $x(t)$ for a few values of m. e.g., $m = 0.1$, $m = 0.01$, $m = 0.001$, etc. Comment on the relation between the solution to the first order ODE and the second order ODE. In what sense is the solution to the first order ODE a good approximation of the second order ODE? Try to plot the $\dot{x}$ vs x for both the second order and the first order equations.

Notes. The first order ODE limit and the full second order ODE are related by a 'singular perturbation'. In a couple of lectures, we will obtain a simple geometric understanding of the relation between the 1st order and 2nd order solutions. Other related key words: 'boundary layers' (in time), multiple time-scales, fast-slow systems.

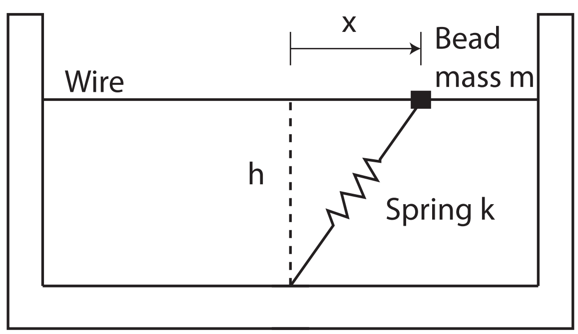

Figure 1: Bead on a wire.

Q4. Reaching zero or infinity. [10pts] Consider the following four differential equations, special cases of $\dot{x} = f(x)$:

$$\dot{x} = kx \quad (1)$$
$$\dot{x} = kx^2 \quad (2)$$
$$\dot{x} = k\,\text{sign}(x) \quad (3)$$
$$\dot{x} = k/x \text{ for } x \neq 0 \text{ and } \dot{x} = 0 \text{ for } x = 0 \quad (4)$$

a) Find all fixed points and their stability using either linear stability analysis or other geometric arguments. Consider all real value cases of k.
b) Note that Equations 3 and 4 are "non-smooth" differential equations, because the $f(x)$ is either continuous but not differentiable (for equation 3) or not even continuous (for equation 4) at $x = 0$. This means that linearization and standard linear stability analysis cannot be used at $x = 0$ for these systems. For such non-smooth functions, write down the necessary conditions for local stability involving one or more of the following (and possibly other terms if necessary):

$$\lim_{x \to 0^+} f(x), \quad \lim_{x \to 0^-} f(x), \quad \lim_{x \to 0^+} \frac{df(x)}{dx}, \quad \lim_{x \to 0^-} \frac{df(x)}{dx}$$

c) For $k > 0$ and initial condition $x(0) = x_0 > 0$, find the closed-form analytical solution for all cases. [4pts]
d) How long does it take for x to "go to infinity" for the four differential equations, for $k > 0$? Equation 2 has what is called a "finite time singularity." [3pts]
e) Now use $k < 0$ and initial condition $x(0) = x_0 > 0$. Compare how quickly each of the equations approach zero. [3pts]

Notes. Equation 3 has some qualitative similarity to mechanical systems with Coulomb friction (if you replace x with speed v). If you were designing a 'control system' to make a stable equilibrium, you may be interested in speeding up the convergence to the stable equilibrium: as you see above 'linear' systems take infinitely long to converge, so sometimes, introducing some non-linearity, even non-smoothness can improve speed of convergence to steady state.