Homework 2, Nonlinear Dynamics, Spring 2016

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Due: Feb 15, 2016

Dynamics in 2D, n-D, bifurcations, phase portraits, linear stability, etc.

You may use `MATLAB` for any question, for instance, to compute eigenvalues while assuming specific numbers for the various parameters, use the symbolic toolbox for analytical solutions, etc., as long as you clearly document what you used.

Q1. Linear system practice. Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{Z} = AZ$ where $Z = [x;\ y]$. Find the characteristic polynomial (whose roots are the eigenvalues), eigenvalues, and eigenvectors of A.

b) Classify the fixed point at origin and sketch the phase portrait by hand qualitatively, but make sure you align any relevant eigenvectors correctly (and this goes for all questions below).

c) Write the general solution for $Z(t)$ in the exponential form in terms of the eigenvalues, the eigenvectors, and unknown constants.

d) For some initial condition, use **matrix exponential** in `MATLAB` to compute the solution to the above ODE and compare with your analytical solution or an `ode45` solution. For matrix exponential, see notes at the end of this HW.

e) Draw a phase portrait using `ode45`, by plotting $x(t)$ vs $y(t)$ from different initial conditions.

Q2. Linear system practice. Consider the system $\dot{x} = x - y$, $\dot{y} = x + y$.

a) Classify the fixed point at origin and sketch the phase portrait qualitatively by hand.

b) Find the general solution for $Z(t)$. Try to write the general solution entirely in terms of real-valued functions.

Q3. Bead on a wire, revisited. Consider the bead on wire system discussed in HW1. In HW1, you used $m = 0$ to reduce the second order to system to a first order system and analyzed the stability of fixed points. Here, you will examine the system again, but now with $m \neq 0$. Assume also $k > 0$ and $c > 0$.

a) Find all the fixed points (equilibria). Consider all parameter regimes, as you did in HW1. Are the fixed points same or different from when $m = 0$?

b) Linearize the nonlinear dynamics about the fixed points by finding the appropriate Jacobians. Determine the stability of the fixed points and classify the nature of the fixed points. Consider all physically plausible parameter ranges you think are significant and give qualitatively different behavior near the fixed points. You will find that there at least four parameter ranges of interest. Does the behavior near the fixed points at high and low damping agree with your physical intuition?

c) Based on your analysis of the fixed points, sketch the phase portraits with axes $y_1 = x$ and $y_2 = \dot{x}$ for the various parameter regimes you considered above. The phase portraits need only be qualitative and approximate.

d) Use ode45 in MATLAB (perhaps using my code allowing to draw phase trajectories from "clicked points") to draw the phase portraits for all the relevant cases considered above by drawing the trajectories for different initial conditions to check your answers for part-c.

e) Fix damping at some value, say $c = 1$. Let's call $h/l_0 = p$. Now, use your analysis above to sketch the regions with qualitatively different behavior in the k- p plane. There is not a standard way to do this, so try to convey the information as best as you can. You will find that the inequality conditions for different behavior can be written in terms of $h/l_0 = p$.

What counts as qualitatively different: Number of fixed points, stability of fixed points, type of fixed points (a spiral or a node), etc.

Suggestion. For the above questions, you do not have to draw the phase portrait for the parameters corresponding to "boundary cases" – for instance, when something goes from stable to unstable, or spiral to node, etc. In other words, consider only parameter regimes defined by "inequality" relations. (Otherwise there would be too many cases).

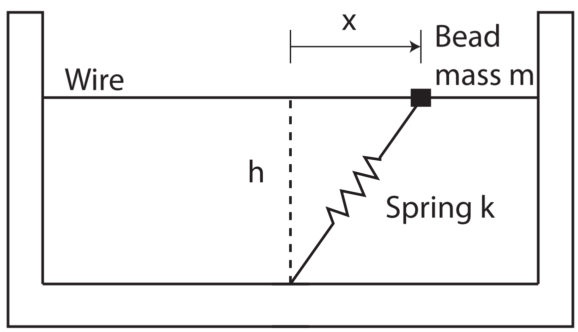

Figure 1: default

Q4. Stabilizing an inverted pendulum. Consider the damped simple pendulum equation from the lectures:

$$\ddot{\theta} + b\dot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

where θ is the angle from the vertical measured anti-clockwise from the bottom position. We know that $\theta = 0$ and $\theta = \pi$ are the only two fixed points, and that $\theta = \pi$ is unstable. To stabilize the inverted position $\theta = \pi$, we add a torsional spring at the pendulum's pivot such that any deviation from $\theta = \pi$ produces a restoring torque proportional to $(\theta - \pi)$, so the equation of motion changes to the following general form:

$$\ddot{\theta} + b\dot{\theta} + \frac{g}{L}\sin\theta + k(\theta - \pi) = 0 \tag{2}$$

a) How high should k be so that inverted pendulum equilibrium $\theta = \pi$ is stable?

b) We already discussed some aspects of the bifurcation diagram for a closely related problem in lecture (specifically, by ignoring inertia and replacing gravity with a different vertical force). Now, using `fsolve` in MATLAB, try to compute all the fixed points and their stability as you change k from 3 to 0.01, say (some meaningful range). For parameters, assume fixed values: $b = 0.2$, $g/L = 1$. Make the MATLAB program generate a bifurcation diagram, with blue dots denoting stable fixed points and red dots denoting unstable fixed points.

Comment: The torsional spring is similar to a linear "proportional feedback controller" with a set point (desired angle) equal to π. And the above question illuminates what "proportional gain" k is necessary for stability and what happens when k is too small (there are too many equilibrium points.). You might also think of this system as a simple model of a big electrical pole in the ground, tied to the ground by cables, which serve as the stabilizing torsional spring.

Q5. A very famous differential equation. Consider the following differential equation:

$$\frac{dx}{dt} = \sigma(y - x) \tag{3}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{4}$$

$$\frac{dz}{dt} = xy - \beta z \tag{5}$$

Determine all fixed points and classify their stability for this set of equations for differential equations. Consider a range of ρ values from 0 to 1 to greater than 1. If necessary, use $\sigma = 10$ and $\beta = 8/3$. You may do this with pencil and paper, use symbolic matlab, or numerically. Your choice, but summarize your answers as a figure or using appropriate description. If you know the type of bifurcation(s), mention that as well.

Extra notes

Matrix exponentials. As you know, the scalar linear equation $\dot{y} = ay$, with $a, y \in \mathbf{R}$ has the solution $y(t) = y(0)e^{at}$. The vector linear equation $\dot{Z} = AZ$, with $A \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^n$, has an analogous analytical solution:

$$Z(t) = e^{At}Z(0)$$

where the matrix $e^{At} \in \mathbf{R}^{n\times n}$ is called the matrix exponential and defined by a matrix Taylor series analogous to that of the scalar exponential. Look up this topic online.