Homework 3, Nonlinear Dynamics, Spring 2016

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Due: Feb 29, 2016, 7 pm
Lyapunov functions, intro to limit cycles, and a forced nonlinear system

Please make physically reasonable assumptions where necessary to fill in gaps in the question set-ups.

Q1. Linear stability is inconclusive for boundary cases. Consider the dynamical system:

$$\ddot{x} + \epsilon \dot{x}^3 + x = 0$$

Clearly $x = 0$ and $\dot{x} = 0$ is an equilibrium point. a) Show that linear stability analysis is inconclusive for this fixed point.
b) Show using Lyapunov functions that $(0, 0)$ is unstable for $\epsilon < 0$.
c) Show using Lyapunov functions whatever method that $(0, 0)$ is an asymptotically stable equilibrium for $\epsilon > 0$.

Q2. Linear stability is inconclusive for boundary cases. Consider the dynamical system:

$$\ddot{x} + c\dot{x} + x^5 = 0$$

with $c > 0$. Clearly $x = 0$ and $\dot{x} = 0$ is an equilibrium point.
a) Show that linear stability analysis say for this fixed point.
b) Show using Lyapunov functions that $(0, 0)$ is asymptotically stable.

Q3. Pendulum with a constant torque. Equations of motion for a pendulum in a plane which is rotating about a vertical axis with angular velocity ω:

$$\frac{d^2x}{dt^2} + \frac{g}{L}\sin x = \frac{T}{mL^2}$$

Find fixed points and bifurcation diagram for $T > 0$. What is the type of bifurcation?

Q4. Stick-slip-like limit cycle oscillations. [16pts] Numerous oscillation phenomena like the creaking of a door hinge, music in some bowed instruments, sudden earthquake fault movement, etc., are attributed to "stick-slip" (see wikipedia perhaps for elaboration). True stick-slip requires static friction (that is, non-zero friction at zero slip rate). Here, for simplicity we will consider friction laws without static friction and try to derive conditions for stick-slip-like oscillatory instabilities. Perhaps in the next HW, we will consider Coulomb friction.

Fig. 1 is the classic set-up for such frictional instabilities. The load point P is moved at a constant velocity v_0 to the right. The mass m is attached to the load point via a spring of stiffness k. The mass m and the table interact via the frictional force F. We will assume that the frictional force F is a pure function of the slip velocity $\dot{y}$. i.e., $F = f(\dot{y})$. Say $x = z - y$, the distance between load point P and m.

a) Derive equations of motion for $x = z - y$ and show that it is of the form. [3pts]

$$\ddot{x} + kx = f(\dot{y}) = f(v_0 - \dot{x}), \tag{1}$$

b) Say $f(\dot{y})$ is an odd function. It is continuous and at least once continuously differentiable. "Steady slip" is defined as when the mass m moves with the same speed as point P ($\dot{z} = \dot{y}$). Derive a condition on f for which steady slip is stable for pull velocity v_0. When is steady slip unstable? [3pts]

c) In some system obeying the above assumptions on f, we have where $x = z - y$ and $f(v) = r_1 v + r_2 v^3 + r_3 v^5$ with $r_1 = 5$, $r_2 = -2$, $r_3 = 0.25$. Figure out for what range of speeds v_0 is steady slip stable. The rest of the questions below use the same f.

d) Using some numerics (say one of the matlab programs posted), draw phase portraits of the system for different values of v_0. Consider different "meaningful ranges" to show qualitatively different phase portraits. In particular, show numerically that when steady slip becomes unstable, we obtain a stable limit cycle.

e) At a given v_0 for which there is a limit cycle, what is the effect of the stiffness k on the limit cycle amplitude? Numerical explorations acceptable. [2pts]

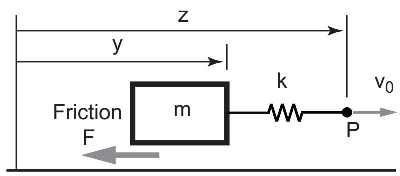

Figure 1: Stick-slip

Q5. Simulate a forced simple pendulum [10 pts]. Consider the forced simple pendulum:

$$\ddot{\theta} + c\,\dot{\theta} + k \sin\theta = A \sin \omega_f t. \tag{2}$$

You will agree that this is a remarkably simple-looking system, just a little bit away from a forced linear oscillator $\ddot{\theta} + c\,\dot{\theta} + k\theta = A \sin \omega_f t$, for which you can write the solutions analytically.

In steady-state, the forced linear oscillator has a periodic motion. Show (numerically) that the forced simple pendulum, on the other hand, can have very complex-looking behavior depending on A, k, ω_f and c. Specifically, find 'some' parameter values such that when you simulate for long enough, the motion is very non-periodic. Use high accuracy ode45 settings.