Homework 5, Nonlinear Dynamics, Spring 2016

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Please solve at least one problem on each topic

Stability of limit cycles and time-periodic systems

Q [10 pts]. Consider again the spring-mass system exhibiting stick-slip-like oscillations (Fig. 1). where $x = z - y$ and $f(v) = r_1 v + r_2 v^3 + r_3 v^5$ with $r_1 = 5$, $r_2 = -2$, $r_3 = 0.25$.

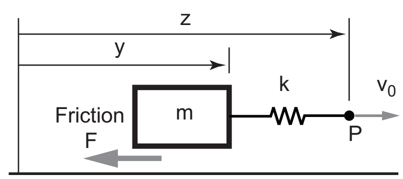

Figure 1: Stick-slip

a) Define a Poincare map by considering the intersection of the trajectory with the horizontal axis $\dot{x} = 0$ as $\dot{x}$ is decreasing (when $\ddot{x} < 0$). Given an initial condition $(x_0, 0)$ on the Poincare section, you can integrate forward in time until the trajectory hits the Poincare section again at, say, $(x_1, 0)$. This defines the Poincare map: $x_{i+1} = P(x_i)$. Using a nonlinear equation solver `fsolve`, find x_0 such that $x_1 = P(x_0) = x_0$. This determines a fixed point of the Poincare map and therefore an initial condition on the Poincare section lying on the limit cycle. For this exercise, you will want to use MATLAB's event detection facility in `ode45`. [10pts]

b) Find a numerical linear approximation to the Poincare map, by determining the Jacobian of the Poincare map $P(x)$. Find the eigenvalues of this Jacobian and show that the fixed point of the Poincare map, and therefore the limit cycle is stable.

Q [10 pts]. Check if $(0, 0)$ is stable for the following linear time-periodic system:

$$\dot{x} = (-3 + 2\sin t)\ x + (0.1\cos 3t)\ y$$

$$\dot{y} = (0.2\sin 2t)\ x + (-2 + \cos 4t)\ y$$

Determine the relevant Floquet multipliers, that is, the eigenvalues of the monodromy matrix. Note that you will generally need to compute this numerically.

Basin (or domain) of attraction

The basin of attraction of an asymptotically stable fixed point is defined as the set of all initial conditions such that trajectories starting at those initial conditions approach the fixed points as $t \to \infty$. The basin of attraction for other "attractors", like an asymptotically stable limit cycle or periodic motion, or an asymptotically stable chaotic attractor is analogously defined. Thus, if a dynamical system has multiple attractors (one or more fixed points and/or limit cycles and/or chaotic motions), each of these attractors will have its own basin of attraction.

The size of the basin of attraction is the "largest deviation" from the fixed point that still ends up at the fixed point. It is one of the measures of 'how stable' a fixed point is (aside from eigenvalues). Given that the basin of is multi-dimensional, with different variables having different units, there is generally not an objective scalar measure of how big it is.

Q. [10 pts] Consider the inverted pendulum with torsional stiffness k and damping c with values such that the inverted state is stable.

$$\ddot{\theta} - \frac{g}{\ell}\sin\theta + c\dot{\theta} + k\theta = 0$$

a) Show that the fixed point $(\theta, \dot{\theta}) = (0, 0)$ is asymptotically stable. Use $g/\ell = 1$, $k = 1.2$, $c = 0.3$. b) Determine the "basin of attraction" of this fixed point. That is, grid up the space of initial conditions $(x, \dot{x})$ around $(0, 0)$ and determine which initial conditions asymptotically approach $(0, 0)$. Use $g/\ell = 1$, $k = 1.2$, $c = 0.3$.

Notes: The above method for finding the basin of attraction is not very computationally efficient for finding the basin of attraction. A more efficient method is based on "cell mapping".

c) How does the size of the basin of attraction change with changes in c or k?

d) Note that the basin of attraction is most useful for nonlinear systems. Think about what the basin of attraction is for an asymptotically stable fixed point of the linear system $\dot{x} = Ax$ with $x \in R^n$.

Q. [10 pts] Figure 2 is a cartoon model of a person standing on two legs. Assume (for simplicity, not necessarily realism) that the two legs are linear springs (with stiffness k, stress-free length ℓ_0) with a linear damper in parallel with coefficient c. The feet A and B are a distance a apart. The legs are massless. The upper body is modeled as a point-mass P of mass m, attached to the legs via torque-free pin joints.

a) Find some values for k, a, ℓ_0 such that a symmetric standing configuration as shown in the figure is stable, with the upper body P, right about point O, the mid-point between A and B. b) For your parameter values from (a), determine the range of initial conditions (x, y) for upper body position P, such that if you start there at rest you end up at the symmetric standing equilibrium position. Note that the state space for this system is four dimensional, and we are just finding a two dimensional slice of the basin of attraction by starting only at rest.

c) Repeat (b) for different values of a. Is it true that the person has a bigger basin of attraction for larger a (wider stance)?

If you'd like plausible parameters (in some non-dimensional units), use $m = 1$, $g = 1$, $k = 10$, $c = 7$, $\ell_0 = 1$. In all the calculations, for simplicity, you may assume that the feet A and B never leave the ground, and always pinned to the ground. This is unrealistic because the feet pinned to the ground will allow tensional leg forces and allow the leg to pull on the ground.

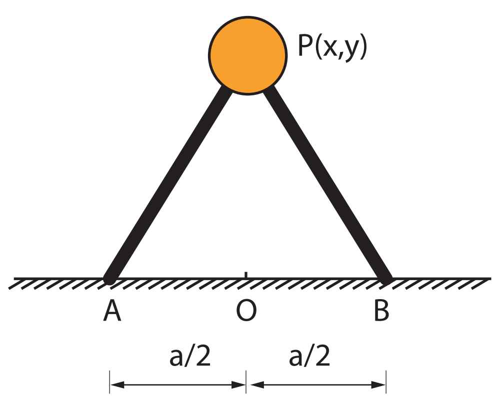

Figure 2: Mass standing on two springy legs.

Q. [10 pts] Consider the following coupled nonlinear equations:

$$\dot{u} = -R^{-1}u + v - v\sqrt{u^2 + v^2} \quad (1)$$

$$\dot{v} = -2R^{-1}v + u\sqrt{u^2 + v^2} \quad (2)$$

a) Determine all the fixed points of this set of equations.
b) Obtain the linearization about the fixed points. Are the fixed points asymptotically stable or unstable? (linear stability analysis)
c) Define what the "basin of attraction" of a fixed point is.
d) Write a simple simulation of the equation. (whatever programming language, matlab is fine).
e) Using (d), determine the basin of attraction of the fixed point $(0,0)$ for different values of R. Say, $R = 10, 5, 1, 0.5, 0.25, 0.1$, etc. Do you see something interesting?
f) Use an analogy for (e). Consider a rectangular box sitting on a table, with base w and height h. The rectangular box is stable for all values $w > 0, h > 0$. Now what happens (as related to the questions above) as you let w becomes smaller $w \to 0$ with fixed h? Explain in words. No equations necessary.
g) Based on what you think about (f), discuss when a system is likely to remain near a stable fixed point in the real world.

Notes: This system is due to Trefethen, who used it to illustrate situations in which one can have asymptotic stability for all parameters but with the basin of attraction becoming very very small for some parameters. Trefethen considered it as a simple model of how turbulence can arise even when laminar flow is asymptotically stable for all Reynolds numbers.

Simulating non-smooth systems

When you integrate the differential equation equations of a non-smooth system, you should avoid integrating "across" the non-smoothness and having if-then statements in your "odefile". These contribute to inaccuracies in the simulation and generally slows the integration way down (this is because most ODE methods are high order methods and assume high-order differentiability in their formulation). So this exercise is to practice integrating non-smooth systems the right way — by using event-detection to stop exactly at the point of non-smoothness and then restart the ODE integration with new equations after the non-smoothness.

Q. Coulomb friction [10 pts]. Consider the same system as in Fig. 1 but now use the Coulomb friction model for $f(\dot{y})$. The classic Coulomb friction is defined as follows, in terms of two regimes:

Sliding. When the slip velocity $\dot{y} = v$ is not equal to zero, the friction force is a constant independent on $|v|$ and is opposite to the direction of v. By our convention,

$$f(v) = \mu_k mg \ \mathrm{sign}(v) \quad \text{if} \quad v \neq 0. \tag{3}$$

No sliding. When $v = 0$, the friction force satisfies the inequality: $-\mu_s mg < f < \mu_s mg$. That is, there is static friction. The friction force equals whatever it takes to maintain $v = 0$ (here, equal to the non-friction external force).

Write a MATLAB ODE integration to simulate this system, by being careful to not integrate over the non-smoothness. You write two ODE files, one for sliding and another for no-sliding. You use event-detection to switch from one regime to the other. You go from sliding to no sliding when the event $v = 0$ happens. And you go from no sliding to sliding when the external force exceeds what can be supported by the static friction.

For some initial conditions with $(y, \dot{y}) = (0, 0)$ and $v_0 = 1$, plot $y(t)$, $\dot{y}(t)$, $x(t)$, and $\dot{x}(t)$. Do two versions of the simulation: One in which the static friction coefficient μ_s equals the kinetic friction coefficient μ_k and one in which $\mu_s > \mu_k$. You should find an extended 'stick' phase in the latter situation.

Advanced note: While $\mu_s > \mu_k$ leads to true stick, this assumption can sometimes lead to weird mechanical effects (see text by Ruina and Pratap).

Q. Chaotic billiards [10 pts]. Do a simulation of some non-smooth system you cook up. e.g., a point particle bouncing inside a rectangular box, where each collision is elastic and there is no gravity. Assume that the motions are all in the same plane, so that you only need to consider 2D planar dynamics in (x, y), the particle's position. There's a huge literature on related problems, with some box shapes giving chaotic motion of the particle.

Perturbation and approximation methods

Q. Multiple scales and super-harmonic resonance for Duffing oscillator. Consider the forced damped Duffing equation:

$$\ddot{x} + \epsilon c\dot{x} + x + \epsilon\alpha x^3 = F\cos(\omega t)$$

with small damping, small nonlinearity, but large forcing (that is, the forcing term does not contain ϵ).

To obtain the frequency response corresponding to a super-harmonic resonance, assume further that the forcing frequency is close to one third the linear natural frequency: that is, $\omega \approx \omega_n/3 = 1/3$, noting that $\omega_n = 1$. Specifically, assume that

$$\omega = 1/3 + \epsilon\omega_1 + \dots$$

Use the method of multiple scales to arrive at a perturbation expansion for steady state $x(t)$. Obtain the frequency response, relating the steady state response amplitude with the frequency ω in the neighborhood of 1/3. Ignore terms of $O(\epsilon^2)$ and just expand everything until $O(\epsilon)$.

Use only two time-scales and not more. In general, the method of multiple scales is well-defined for two time-scales, and doing a higher order analysis allows too much freedom, so that it results in ambiguities. See book by Bender and Orszag for some details.

Q. Raleigh oscillator. The Raleigh oscillator is the following ODE, which (for instance) arises naturally in simple electrical circuits with diodes.

$$\ddot{y} + y = \epsilon\left(\dot{y} - \frac{\dot{y}^3}{3}\right)$$

This system has an unstable focus fixed point at the origin and stable limit cycle. Determine equations that describe the approach to this limit cycle, as well as the amplitude of the limit cycle.

You can use the Method of Multiple Scales or the Method of Averaging.

Q. Lindstedt Poincare method for unforced undamped Duffing with quadratic nonlinearity.

$$\ddot{x} + x + \epsilon\alpha x^3 + \epsilon\beta x^2 = 0$$

For simplicity, set $\alpha = 0$ and $\beta \neq 0$, and use the Lindstedt-Poincare method to determine the amplitude of free oscillations as a function of frequency. Then, try $\alpha \neq 0$.