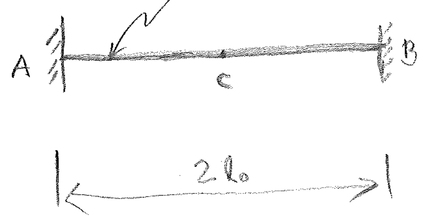
SIMPLE EXAMPLE OF A NONLINEAR SPRING

A taut string as a nonlinear spring (rubber band)

Consider a rubber band / piece of string stretched between rigid walls A and B. C is the midpoint of AB.



Rest / stress-free length of rubber band $= 2l_s$

Initial stretched length of rubber band $= 2l_0$

$= |AB|$

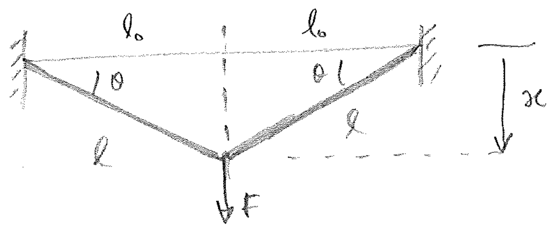
$= 2|AC|$

Initial tension in rubber band $= T_0 = \underset{\substack{\uparrow \\ \text{axial spring constant}}}{k}\,\underbrace{(2l_0 - 2l_s)}_{\text{stretch}}$

Note: $l_0 > l_s$

Now, say we pull the rubber band downward at point C with downward force F, resulting in a vertical displacement x.

What is the $F(x)$? ← NONLINEAR SPRING FORCE

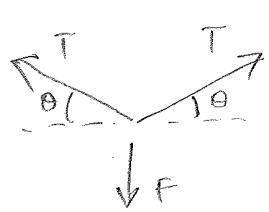


Say the tension in the rubber band is T.

The new length of the rubber band is $2l$.

$$T = k(2l - 2l_s) \quad \text{— (1)}$$

Free body diagram.



Force balance in vertical direction $\Big\}$ $2T\sin\theta = F$ — (2)

Geometry

$$l\cos\theta = l_0$$

$$l = \frac{l_0}{\cos\theta}, \qquad l = \sqrt{l_0^2 + x^2} \quad -(3)$$

$$\sin\theta = x/\sqrt{x^2 + l_0^2} \quad -(4)$$

Use (1), (3), (4) in (2),

$$F = 2T\sin\theta = 2k(2l - 2l_s)\sin\theta$$

$$= 4k(l - l_s)\sin\theta$$

$$\boxed{F(x) = 4k\left[\sqrt{l_0^2 + x^2} - l_s\right]\frac{x}{\sqrt{x^2 + l_0^2}}}$$

Thus the vertical force is some non-linear function of x.

Exercise : (1) Find a linear approximation to $F(x)$ applicable when x is "small" (when $x \ll l_0$.)

(2) What happens to the above linear approximation when $l_0 = l_s$? In this case, find the lowest order polynomial approximation to $F(x)$ of the form cx^α.

Hint: You can use Taylor series of $F(x)$ about $x = 0$.

(or) use ad hoc approximations like $\sqrt{l_0^2 + x^2} \approx l_0$

$$\approx l_0\left(1 + \frac{1}{2}x^2/l_0^2\right), \text{ etc}$$

TAYLOR SERIES AT $x = 0$

Near $x = 0$

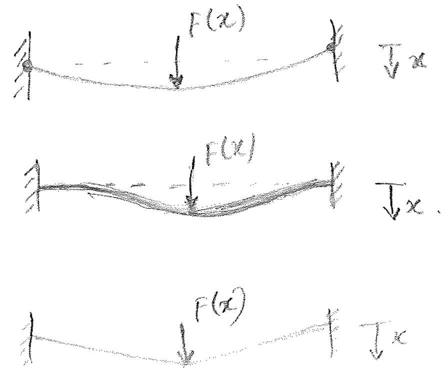
$$f(x) = \underbrace{\underbrace{f(0) + x f'(0)}_{\text{LINEAR APPROXIMATION}} + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)}_{\text{CUBIC APPROXIMATION}} + \ldots$$

LECTURES 3 + 4A

Cubic nonlinearity and Symmetry

Consider the following systems:

For large x, $F(x)$ is some nonlinear function of x.



pinned-pinned beam

fixed-fixed beam

taut rubber band / string.

We can show that a low-order polynomial (Taylor) approximation is cubic, with no quadratic term. Why?

First, note that for all these systems, the following property holds by symmetry.

$$F(-x) = -F(x). \quad — (0)$$

That is, if you apply a force $-F(x)$, you get a displacement $-x$ and the force is $F(-x)$ by definition [So $F(-x) = -F(x)$]

That is, $F(x)$ is an "odd" function.

Say $F(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3$

We know $F(0) = 0$. $\Rightarrow \boxed{a_0 = 0}$

$F(x) = -F(-x) \Rightarrow$ $\cancel{a_0} + \cancel{a_1 x} + a_2 x^2 + \cancel{a_3 x^3} = -(\cancel{a_0} - \cancel{a_1 x} + a_2 x^2 - \cancel{a_3 x^3})$

$$2a_2 x^2 = 0$$

$$\boxed{a_2 = 0} \Rightarrow \boxed{F(x) = a_1 x + a_3 x^3}$$

alternate proof: $F(x) = -F(-x)$
$F'(x) = +F'(-x)$
$F''(x) = -F''(-x) \Rightarrow 2F''(0) = 0$ $\boxed{F''(0) = 0}$

If Eqn (0) ceases to be true, then we might start getting both the a_0 and the $a_2 x^2$ terms. That is, if the up-down symmetry of the system behavior is broken somehow:

Eg., with an initial strain. i.e., $F = 0$ corresponding to $x \neq 0$ (or) $x = 0$ corresponding to $F \neq 0$, etc.

Ex: More generally, if (0) is true, can you show that the only Taylor terms are odd powers?