A taut string as a nonlinear spring

Consider a rubber band / piece of string stretched between rigid walls A and B. C is the midpoint of AB.

Rest/stress-free length of rubber band = \( 2l_0 \)

Initial stretched length of rubber band = \( 2l_0 \)

Initial tension in rubber band = \( T_0 = k(2l_0 - 2l_s) \)

Now, say we pull the rubber band downward at point C with downward force \( F \), resulting in a vertical displacement \( x \).

What is the \( F(x) \)?

Say the tension in the rubber band is \( T \).

The new length of the rubber band is \( 2l_s \).

\[
T = k(2l_s - 2l_s) = 0
\]

Free body diagram:

\[
2T \sin \theta = F
\]
Geometry

\[ l \cos \theta = l_0 \]
\[ l = \frac{l_0}{\cos \theta} \]
\[ l = \sqrt{l_0^2 + x^2} \]
\[ \sin \theta = x/\sqrt{x^2 + l_0^2} \]

Use (1), (3), (4) in (2),

\[ F = 2T \sin \theta = 2k(2l - 2l_0) \sin \theta \]
\[ = 4k(l - l_0) \sin \theta \]

\[ F(x) = 4k \left[ \sqrt{l_0^2 + x^2} - l_0 \right] \frac{x}{\sqrt{x^2 + l_0^2}} \]

Thus the force is some non-linear function of \( x \).

Exercise: (1) Find a linear approximation to \( F(x) \) applicable when \( x \) is "small" (when \( x \ll l_0 \)).

(2) What happens to the above linear approximation when \( l_0 = l_0 \)? In this case, find the lowest order polynomial approximation to \( F(x) \) of the form \( cx^2 \).

Hint: You can use Taylor series of \( F(x) \) about \( x = 0 \).

Near \( x = 0 \)

TAYLOR SERIES AT \( x = 0 \)

\[ f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \]

LINEAR APPROXIMATION

CUBIC APPROXIMATION
Cubic nonlinearity and symmetry

Consider the following systems:

For large \( x \), \( F(x) \) is some nonlinear function of \( x \).

We can show that a low-order polynomial (Taylor) approximation is cubic, with no quadratic term. Why?

First, note that for all these systems, the following property holds by symmetry:

\[ F(-x) = -F(x). \] \( \text{(0)} \)

That is, if you apply a force \(-F(x)\), you get a displacement \(-x\).

And the force is \( F(-x) \) by definition. \([\text{So } F(-x) = -F(x)]\)

That is, \( F(x) \) is an "odd" function.

Say \( F(x) = a_1 x + a_2 x^2 + a_3 x^3 \)

We know \( F(0) = 0 \), \( \Rightarrow \) \( a_0 = 0 \)

\( F(x) = -F(-x) \)

alternate proof:
\( F(x) = -F(-x) \)
\( F'(x) = -F'(-x) \)
\( F''(x) = -F''(-x) = 2F''(0) = 0 \)

If Eq. (0) were to be true, then we might start getting both the \( a_0 \) and the \( a_2 x^2 \) terms. That is, if the up-down symmetry of the system behavior is broken, then:

Eq. with an initial strain. \( \Rightarrow \) \( F = 0 \) corresponding to \( x \neq 0 \)

(Or \( x = 0 \) corresponding to \( F = 0 \), etc.)

Ex: More generally, if (0) is true, can you show that the only Taylor terms are odd powers?