REDUCING A HIGH ORDER ODE TO A SET OF FIRST ORDER ODEs

This is always possible, by introducing appropriate new variables.

Example 1 : $m\ddot{x} + f(x) = q(t)$ — (1)

→ Introduce new variables:

$x_1 = x$ — (2)

$x_2 = \dot{x}$ — (3)

→ Then, the ODEs governing x_1 and x_2 are:

(4) $\dot{x}_1 = x_2$ from (3)

$\dot{x}_2 = \ddot{x} = \dfrac{q(t) - f(x_1)}{m}$

(5) $\dot{x}_2 = \dfrac{q(t) - f(x_1)}{m}$.

(4) and (5) are coupled first order ODEs governing x_1 and x_2 equivalent to the 2nd order ODE (1).

Example 2

$$\left.\begin{aligned} m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 &= q_1 \\ m_{21}\ddot{\theta}_1 + m_{22}\ddot{\theta}_2 &= q_2 \end{aligned}\right] \quad (6)$$

Introduce new variables for all the θ's and $\dot{\theta}$'s.

So
$$\left.\begin{aligned} x_1 &= \theta_1 & x_3 &= \theta_2 \\ x_2 &= \dot{\theta}_1 & x_4 &= \dot{\theta}_2 \end{aligned}\right] \quad (7)$$

Using (6) and (7), we might write the first order ODEs for x_1 to x_4.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_3 &= x_4 \end{aligned}\right] \quad (8)$$

$$\left.\begin{aligned} m_{11}\dot{x}_2 + m_{12}\dot{x}_4 &= q_1 \\ m_{21}\dot{x}_2 + m_{22}\dot{x}_4 &= q_2 \end{aligned}\right] \Rightarrow \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

$$\text{So } \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \quad (9)$$

(8) and (9) are 4 first order ODEs in x_1 to x_4 equivalent to the 2 second order ODEs in (6).

Example 3.

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}x}{dt^{n-2}} + \dots + a_1\frac{dx}{dt} + a_0 = 0.$$

New variables : $x_1 = x$, $x_2 = \dfrac{dx}{dt}$, ..., $x_{i+1} = \dfrac{d^i x}{dt^i}$ — x_i where $i = 1$ to n. 'n' variables. where $x_n = \dfrac{d^{n-1}x}{dt^{n-1}}$.

New 1st order equations for x_1 to x_n :

For $i = 1$ to $n-1$, $\dot{x}_i = x_{i+1}$, and $\dot{x}_n = -a_{n-1}x_n - a_{n-2}x_{n-1} + \dots - a_1 x_2 - a_0$

Equivalence of 2 forms of linear homogeneous equations.

Forward : One 2^{nd} order ODE :

$$m\ddot{x} + c\dot{x} + kx = 0$$

We can convert this ODE to 2 coupled 1^{st} order ODEs.
by introducing new variables:

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx}{dt} \end{aligned}\right] \Rightarrow \quad \frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = \frac{-(cx_2 + kx_1)}{m}.$$

$$Y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \quad \frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = AY.$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \Longrightarrow \boxed{\dot{Y} = AY}$$

Backward : Now consider the most general form of $\dot{Y} = AY$. We now show that this can be reduced to a single 2^{nd} order equation.

$$\dot{Y} = AY. \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(1) $\dfrac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2$

(2) $\dfrac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2$

$$(1) \Rightarrow \frac{d^2 y_1}{dt^2} = a_{11}\frac{dy_1}{dt} + a_{12}\frac{dy_2}{dt} = a_{11}\frac{dy_1}{dt} + a_{12}(a_{21}y_1 + a_{22}y_2)$$

$$= a_{11}\frac{dy_1}{dt} + a_{12}a_{21}y_1 + a_{12}a_{22}y_2$$

$$= a_{11}\frac{dy_1}{dt} + a_{12}a_{21}y_1 + a_{12}a_{22}\left[\frac{1}{a_{12}}\left(\frac{dy_1}{dt} - a_{11}y_1\right)\right]$$

Simplifying:

$$\frac{d^2y_1}{dt^2} = a_{11}\frac{dy_1}{dt} + a_{22}\frac{dy_1}{dt} + y_1\left(a_{12}a_{21} - a_{11}a_{22}\right).$$

$$\boxed{\frac{d^2y_1}{dt^2} - \left(a_{11}+a_{22}\right)\frac{dy_1}{dt} + \left(a_{11}a_{22} - a_{12}a_{21}\right)y_1 = 0}$$

ie, we have shown $\boxed{\dot{Y} = AY} \Rightarrow$ 2nd order ODE in y_1.

We can also show

$$\frac{d^2y_2}{dt^2} - \left(a_{11}+a_{22}\right)\frac{dy_2}{dt} + \left(a_{11}a_{22} - a_{12}a_{21}\right)y_2 = 0$$

That is, y_1 and y_2 follow the same ODE.

Indeed, <u>Any</u> linear combination $z = k_1y_1 + k_2y_2$ also follows the same ODE. $\ddot{z} - (a_{11}+a_{22})\dot{z} + (a_{11}a_{22} - a_{12}a_{21})z = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \qquad a_{11} + a_{22} = \underline{\text{Trace}}(A) = \text{Sum of diagonal elements.}$$

$$a_{11}a_{22} - a_{12}a_{21} = \underline{\text{Determinant}}(A).$$

So we can write the equivalent ODE as:

$$\boxed{\frac{d^2y_1}{dt^2} - \text{Trace}(A)\frac{dy_1}{dt} + \text{Det}(A)\,y_1 = 0}$$

<u>More generally</u>, it can be shown that if $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, then there is an equivalence between

$$\boxed{\dot{Y} = AY} \quad \text{and} \quad \boxed{y_1^{(n)} + b_n y_1^{(n-1)} + b_{n-1}y_1^{(n-2)} + \dots + b_2 y^{(1)} + b_1 = 0}$$

<u>Exercise</u>: Find the relation between A and b_i.

ASIDE : What about nonlinear systems?

Given $\ddot{x} + f(x, \dot{x}) = 0$,

Introducing , $\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right]$ $\left.\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1, x_2) \end{array}\right] \Rightarrow \dfrac{dY}{dt} = f(Y).$

This is a useful reduction and always possible. We will use this reduction over and over again throughout this course.

Given $\dot{y}_1 = f_1(y_1, y_2)$ — (1)

$\dot{y}_2 = f_2(y_1, y_2)$ — (2)

$$\frac{d^2 y_1}{dt^2} = \frac{\partial f_1}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial f_2}{\partial y_2}\frac{dy_2}{dt} = \frac{\partial f_1(y_1, y_2)}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial f_2(y_1, y_2)}{\partial y_2} f_2(y_1, y_2) \quad (3)$$

Say we can invert (1) so that $y_2 = g\left(y_1, \frac{dy_1}{dt}\right)$ Then we can plug this in (3) to get a single 2nd order ODE in y_1 (nonlinear).

This reduction is not very useful, hard to do in practise (because of the nonlinear inversion which may not always be possible). We will never use such a reduction.