A first order system as a singular limit of a second order system with low inertia

In the first few lectures,

we considered the second order equation

$$m\ddot{x} + c\dot{x} + f(x) = 0 \qquad -(1)$$

in the limit of $m \to 0$ (or at least very small m).

We ignored the $m\ddot{x}$ term to obtain

$$c\dot{x} + f(x) = 0$$

$$\dot{x} = \frac{-f(x)}{c} \qquad -(2)$$

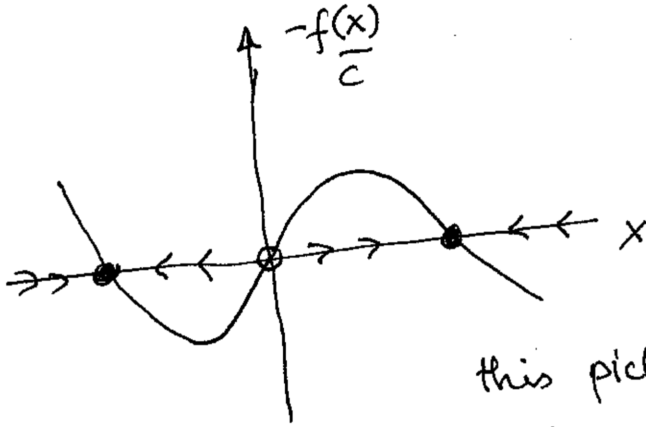
We noted that (1) required 2 scalar initial conditions $x(0), \dot{x}(0)$

but (2) requires only $x(0)$.

What is the nature of the relationship between solutions to (1) and (2)?

Solution to (2)

Say we have this $\frac{-f(x)}{c}$ with 3 fixed points.



this picture describes what happens for any initial $x(0)$.

Solution to ①

Equation ① requires 2 scalar initial conditions $x(0)$ and $\dot{x}(0)$. Let us rewrite the equation ① as 2 first order equations:

$$\dot{x} = v \quad \text{velocity.}$$
$$\dot{v} = \frac{-(cv + f(x))}{m} \quad \text{acceleration.}$$

Consider some initial condition $x(0)$, $v(0)$.

if $\dot{v}(0) = \dfrac{-[cv(0) + f(x(0))]}{m}$

is HUGE if m is very small

as long as $cv(0) + f(x(0)) \neq 0$

if $cv + f(x) > 0$,

$\dot{v} =$ HUGE and < 0

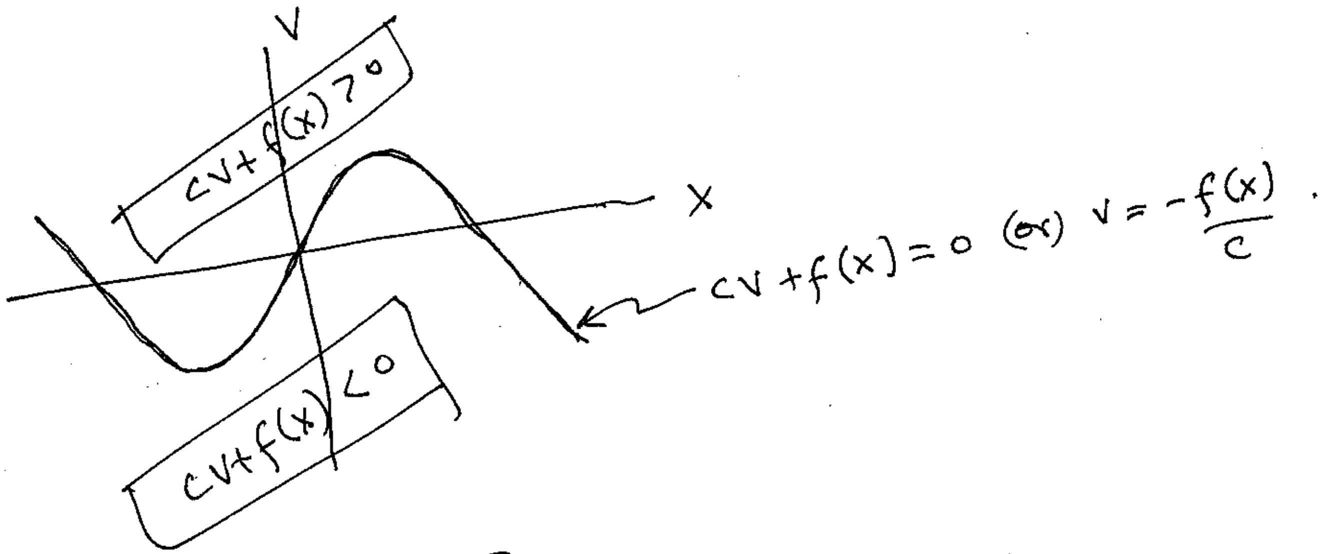
if $cv + f(x) < 0$

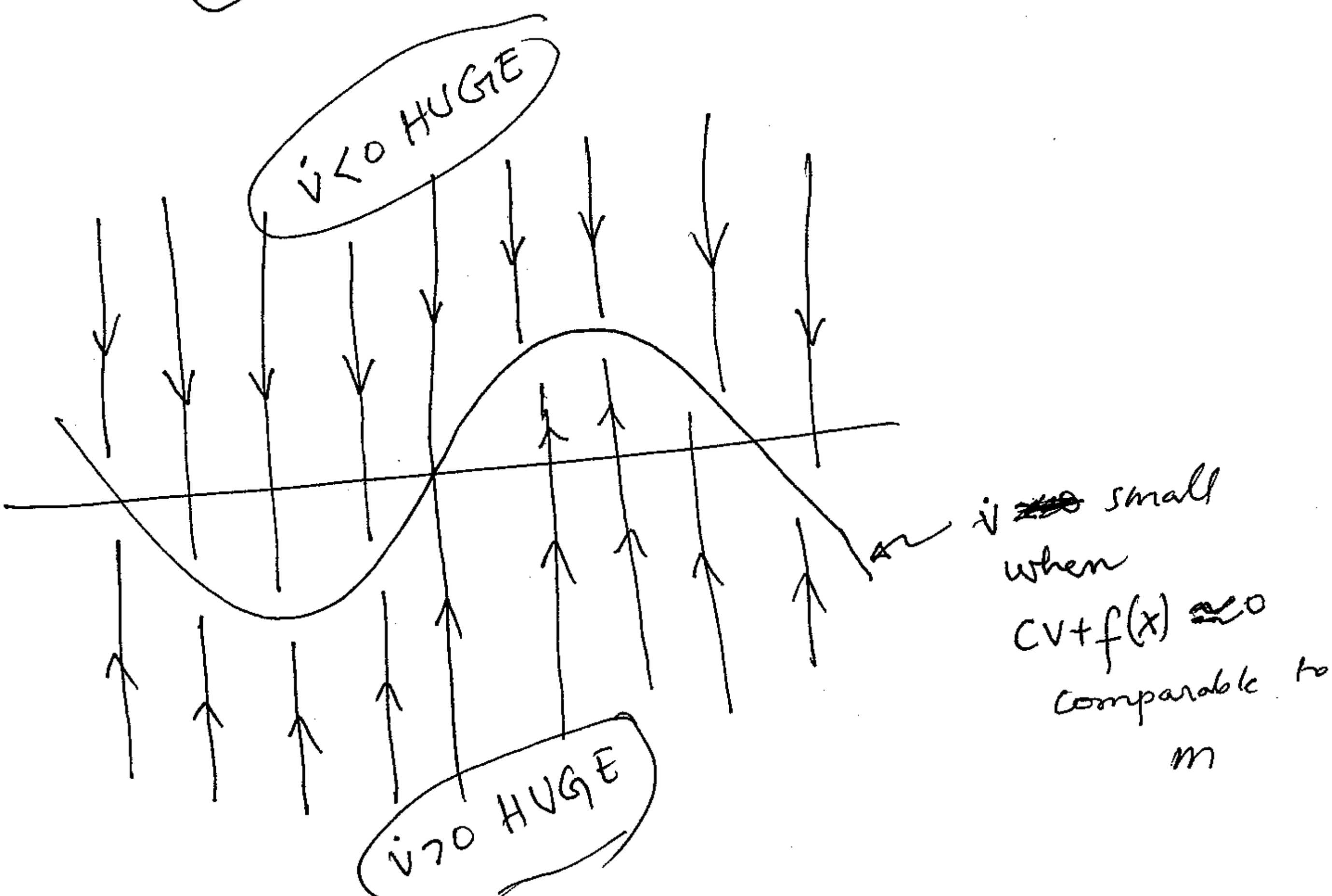
$\dot{v} =$ HUGE and > 0.

Let's see what this implies in pictures.

first, we divide the phase space into 3 regions.

$$cv + f(x) > 0$$
$$cv + f(x) = 0$$
$$cv + f(x) < 0.$$





Thus, when m is VERY small we get HUGE accelerations $\dot{v}$ that drives the state toward the $cv+f(x)$ curve VERY FAST.

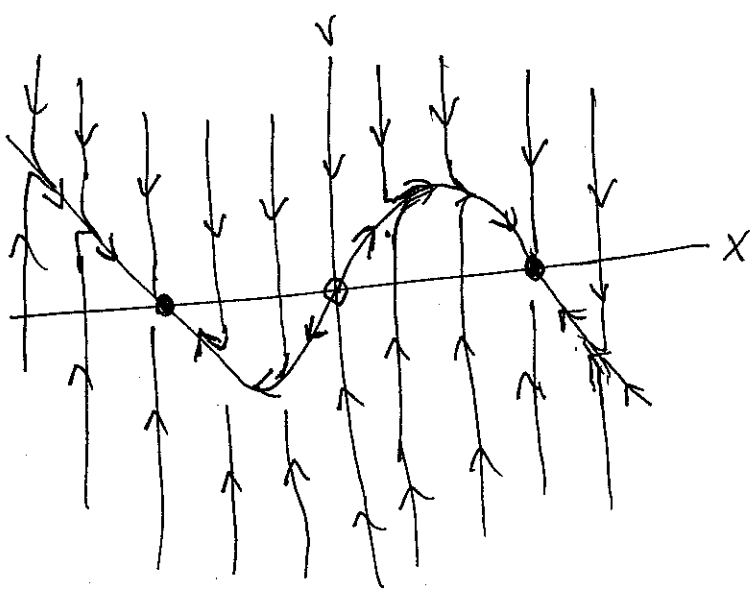
Once we are very close to the $cv+f(x)$ curve

$\dot{v}$ becomes non-HUGE.

and the dynamics are dominated by the equation $cv+f(x)=0$

$$v = -\frac{f(x)}{c} \quad \text{(or)} \quad \dot{x} = -\frac{f(x)}{c}$$

ie, by the first order ODE. The system crawls along the $cv+f(x)=0$ toward the appropriate fixed point. shown below.



you can check this phase portrait numerically with $f(x) = x - kx^3$

So this is how the 1^{st} order equation & the 2^{nd} order equation are related when $m \approx 0$.

When $m \approx 0$, the 2^{nd} order equation has a solution with 2 pieces

(1) a <u>very fast initial transient</u> with fast acceleration that brings the system quickly to

$$cv + f(x) = 0$$

$$\text{(or)} \quad v = -f(x)/c.$$

(2) a <u>slow transient</u> that, to a very good approximation, obeys the first order ODE

$$\dot{x} = -f(x)/c.$$

The 1^{st} order ~~ode~~ ODE and the second order ODE agree except during the brief initial transient.

~ initial brief fast transient is called (sometimes) a boundary layer (in time) which is analogous to the spatial boundary layer in fluid mechanics.

~ this system, having 2 time-scales that are vastly different, are sometimes called

slow-fast systems.

(another example is an earthquake: slow creep over many years + fast shaking over a few seconds).
we'll see more examples later on.)