Stability defined precisely

Lyapunov stability

Premise : Consider the ordinary differential equation

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ (ODE with n scalar state variables)

f is a continuous function.

x^* is an equilibrium point, so that $f(x^*) = 0$.

Definition : Lyapunov stability.

x^* is Lyapunov stable if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$, then for all $t \geq 0$, we have $\|x(t) - x^*\| < \epsilon$.

What does this mean?

→ $\|x(0) - x^*\| < \delta$

means

the initial condition $x(0)$ is within a distance δ of the equilibrium point x^*.

→ $\|x(t) - x^*\| < \epsilon$ for all $t \geq 0$

means

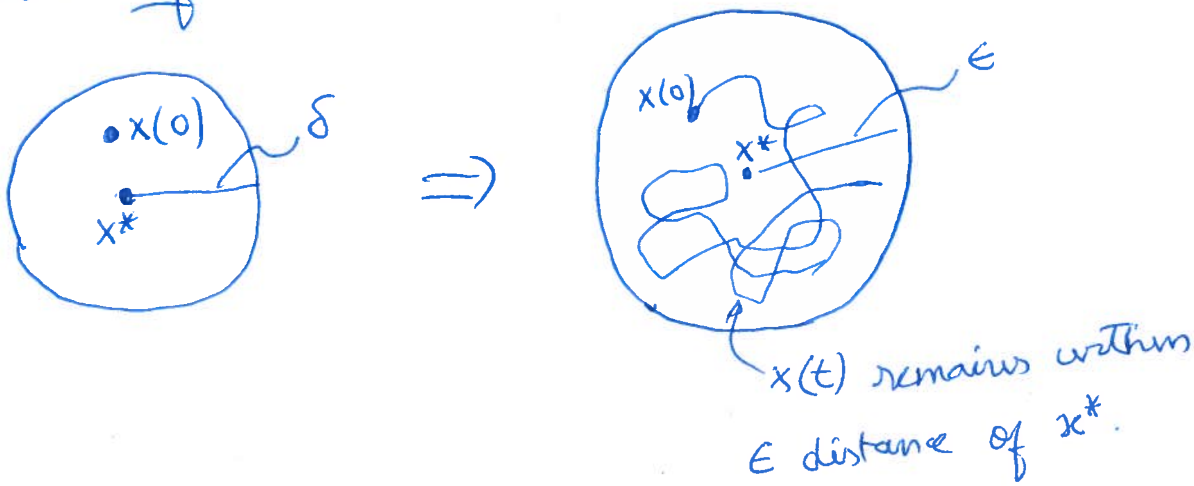
the $x(t)$ is within a distance ϵ of the equilibrium point x^* for all time.

Lyapunov stable <u>if</u>

someone picks an ϵ

and you can always pick a corresponding δ

so that <u>if</u> the initial condition is within δ of x^*.



More informally,

you can always start close enough to x^* (within distance $\delta > 0$) so that the trajectory $x(t)$ always remains close to x^* (within <u>whatever</u> distance $\epsilon > 0$).

<u>Examples</u>

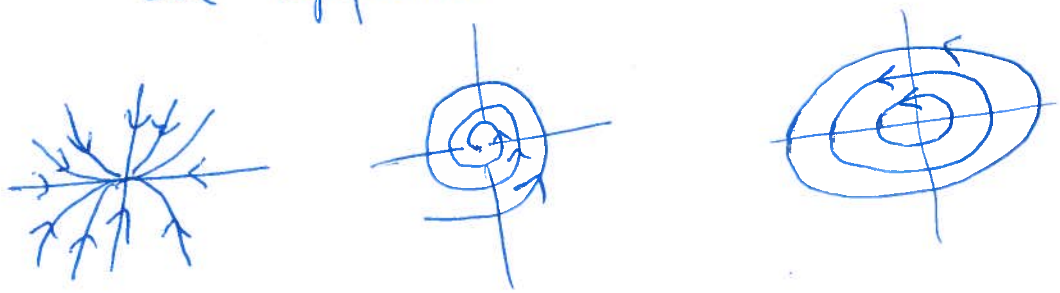
(1) All eigenvalues of the Jacobian of $f(x)$ at x^* have negative real parts.

$\Rightarrow$ <u>Lyapunov stable</u>.

(2) in 2D $(x \in \mathbb{R}^2)$ linear systems.

stable spiral, stable node, <u>center</u>

all Lyapunov stable.



~~Not~~ If even one eigenvalue of Jacobian of $f(x)$ at x^* has a positive real part

$\Rightarrow$ not Lyapunov stable.

(Lyapunov unstable).

If some eigenvalues of Jacobian have negative real parts and some have <u>zero</u> real parts; the eq. point x^* can be either Lyapunov stable or unstable.

Need to do a nonlinear analysis to determine stability.

Asymptotically stable fixed point

<u>Definition</u>. x^* is asymptotically stable if it is Lyapunov stable <u>and</u> there exists a $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$, then

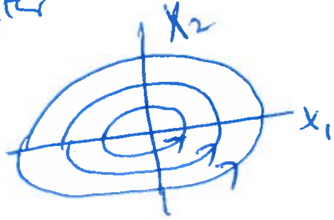
$$\lim_{t \to \infty} \|x(t) - x^*\| = 0$$

<u>More informally</u>, $x(t) \to x^*$ as $t \to \infty$. ~~(and does not~~
if you start close enough to x^*.

<u>Examples</u>

(1) All eigenvalues of Jacobian of $f(x)$ at x^* have negative real parts
$\Rightarrow x^*$ is asymptotically stable.

(2) center



is Lyapunov stable
but NOT asymptotically stable

(3) finite time stability is a special case of asymptotically stable. if $x(t) \to x^*$ in finite

time, it is also true that $x(t) \to x^*$ as $t \to \infty$.

Exponential stability

x^* is exponentially stable if it is asymptotically stable and there exists α, β, δ such that

if $\|x(0) - x^*\| < \delta$, then

$$\|x(t) - x^*\| \leq \alpha \, \|x(0) - x^*\| \, e^{-\beta t} \quad \text{for all } t \geq 0$$

This just bounds the rate of approach as at least as fast as an exponential decay.

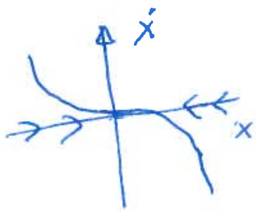
Examples.

(1) If all eigenvalues of the Jacobian of $f(x)$ at x^* have negative real parts
$\Rightarrow$ exponentially stable.

(2) $\dot{x} = -x^3$

$x^* = 0$ is Lyapunov stable, asymptotically stable

But NOT exponentially stable.
(approach to x^* is slower than exponential).



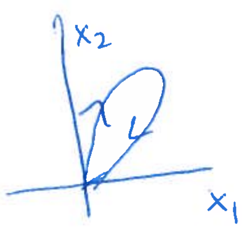
Further remarks.

(1) exponentially stable $\Rightarrow$ asymptotically stable $\Rightarrow$ Lyapunov stable.

(2) There are weird examples which have the property that $x(t) \to x^*$ as $t \to \infty$ but not Lyapunov stable. eg. Hahn's example (1967 book pg. 191)

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2+x_2^2)\left(1+(x_1^2+x_2^2)^2\right)} \quad \& \quad \dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2+x_2^2)\left[1+(x_1^2+x_2^2)^2\right]}$$

(originally due to Vinograd).



origin is a fixed pt. & attractive. But there's a loop that starts & ends at 0. $\Rightarrow$ $\boxed{\text{not Lyapunov stable}}$

because we cannot find a δ for every ϵ.

even though there is a neighborhood around 0 such that if you start from there $x(t) \to 0$ as $t \to \infty$.

-(Such loops that start and end at a fixed point are called homoclinic orbits)-