Lyapunov functions to establish Lyapunov stability

Example 1

Lets start with a simple example.

$$m\ddot{x} + Kx + c\dot{x} = 0 \qquad k, m, c > 0$$

$$\dot{x} = v$$

$$\dot{v} = \frac{-Kx - cv}{m}$$

Is $\begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ stable?

We know it is stable. But lets try to figure this out another way.

Consider the total energy $E(x, v)$

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 .$$

How does the energy change as the system evolves?

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial v}\frac{dv}{dt}$$

$$\frac{\partial E}{\partial x} = \frac{2kx}{2} = kx \quad , \quad \frac{\partial E}{\partial v} = \frac{2mv}{2} = mv$$

$$\frac{dx}{dt} = v \quad , \quad \frac{dv}{dt} = \frac{-kx - cv}{m} .$$

$$\Rightarrow \quad \frac{dE}{dt} = (kx)v + (mv)\left(-\frac{kx}{m} - \frac{cv}{m}\right).$$

$$= \cancel{kxv} - \cancel{kxv} - cv^2 = -cv^2 .$$

$$\boxed{\frac{dE}{dt} = -cv^2} \Rightarrow \frac{dE}{dt} < 0 \quad \text{as long as } v \neq 0 .$$

<u>What does this mean?</u>

Note that $E(0,0) = 0$.

$E(x,v) > 0$ for $x \neq 0$ & $v \neq 0$.

And $\frac{dE}{dt} < 0$ when $v \neq 0$

Let us for now ignore the few points (when v = 0), when dE/dt = 0

$\Rightarrow$ wherever we start, E keeps decreasing. It will eventually end up at $E = 0$

$\Rightarrow$ we end up at $\begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ because this is the only point at which $E = 0$.

Okay, in the argument on the previous page, we ignored the possibility that dE/dt could be zero. To make the argument complete, we have to address this issue.

In this example, as we just saw, $dE/dt = 0$ only when $v = 0$.

IF $v \neq 0$, $dE/dt < 0$. That is E decreases.

Consider a moment in a trajectory when $v = 0$. When $v = 0$,

$$\dot{v} = -kx/m.$$

This means that as long as $x \neq 0$, $\dot{v} \neq 0$ and v immediately becomes non-zero.

This means that v cannot persist being 0 (unless we are already at the fixed point). So $dE/dt = 0$ cannot persist for an extended time duration > 0.

Thus, wherever we start (not already at the fixed point), the system spends almost all the time with $dE/dt < 0$, while passing through dE/dt at a discrete set of times. This means that dE/dt will be < 0 on average. So $E \to 0$. This, in turn, implies that $(x, v) \to (0, 0)$.

Example 2

$$\frac{dx}{dt} = -kx$$

Consider the function $V(x) = x^2$, say.

How does V change as x evolves?

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$= (2x) \cdot (-kx) = -2kx^2 < 0 \text{ if } x \neq 0$$

$\frac{dV}{dt} < 0$ if $x \neq 0$.

what does this mean?

wherever we start with $x \neq 0$

V keeps decreasing $\left(\frac{dV}{dt} < 0\right)$

V eventually gets to $V = 0$.

But $V = 0$ only when $x^* = 0$

$\Rightarrow$ we eventually get to $x^* = 0$.

Lyapunov generalized these ideas (in these two examples) to prove stability using how a function $V(x)$ changes as $x(t)$ evolves.

↳ called a Lyapunov function.

Lyapunov's method for establishing
Lyapunov stability
using "Lyapunov functions"

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$.
$0 = x^*$ is an equilibrium point so that $f(x^*) = 0$.

$0 = x^*$ is Lypunov stable if

we can find a function $V(x) : \underbrace{\mathbb{R}^n \to \mathbb{R}}$ — $V(x)$ is a scalar

such that, in a neighborhood containing $x^* = 0$,

→ $V(x) = 0$ if and only if $x = 0$

→ $V(x) > 0$ if and only if $x \neq 0$

→ $\frac{dV}{dt}$ along a trajectory $x(t)$ is ≤ 0

ie, V decreases as you move along a solution $x(t)$

if $\frac{dV}{dt} < 0$ | if V keeps decreasing, it eventually ends up at $V(x) = 0 \Rightarrow x(t) \to x^* = 0$.

Remarks.

(1) $V(x)$ that satisfies all the above conditions and helps establish Lyapunov stability is called a Lyapunov function.

(2) $$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \cdots$$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x).$$

(3) if $$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \leq 0$$

$\Rightarrow$ Lyapunov stable.

if $$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0$$

$\Rightarrow$ asymptotic stable.

(4) ~~Note~~

Lypunov method applied to linear systems.

$\dot{x} = Ax$. with $x \in \mathbb{R}^n$.

We use a quadratic Lyapunov function

$$V(x) = x^T P x$$

where P is positive definite and symmetric.

$$\left\{ \begin{array}{l} \text{all eigenvalues are real \& positive} \\ x^T P x \geq 0 \text{ for every } x \neq 0 \\ x^T P x = 0 \iff x = 0 \end{array} \right.$$

$$\frac{dV}{dt} = \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P (Ax) = x^T A^T P x + x^T P A x$$

$$= x^T \left(A^T P + PA \right) x.$$

if $\frac{dV}{dt} \leq 0 \Rightarrow$ Lyapunov stable.

if we can find such a P.

Condition for $\frac{dV}{dt} \leq 0$

is $A^T P + PA$ is negative definite.

i.e. $x^T (A^T P + PA) x \leq 0$ for every x.

Exercise

- Use an energy-like Lyapunov function to prove the asymptotic Lyapunov stability of the fixed point $\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for the following ODE.

$$m\ddot{x} + c\dot{x} + kx^n = 0$$

where n is odd and $n \geq 3$.

- Show that linear stability analysis <u>does</u> not work for this case. i.e., it is not <u>conclusive</u>.

LaSalle's Invariance Principle

The basic Lyapunov theorems say:

- $dV/dt \leq 0$ implies Lyapunov stability.
- $dV/dt < 0$ implies asymptotic stability.

where V is the Lyapunov function, satisfying all the other necessary properties.

In many situations, it is possible to infer **asymptotic stability** even when we can only show $dV/dt \leq 0$ when not at the fixed point.

The basic idea is the same in our first example. We have to show that $dV/dt = 0$ cannot persist for an extended period of time, and any trajectory not already at the fixed point will almost always have $dV/dt < 0$, so that $V \to 0$.

This idea is formalized by the so-called **LaSalle's invariance principle**, stated as follows:

First, determine all 'complete trajectories' of $\dot{x} = f(x)$ at which $dV/dt = 0$ identically. One such trajectory is the fixed point 0. Are there any other trajectories? If there are no other complete trajectories satisfying $dV/dt = 0$, then $dV/dt \leq 0$ implies **asymptotic stability**.

How to show that there are no other complete trajectories with $dV/dt = 0$? As in example 1, we can show that if $dV/dt = 0$ at some point (not already at the fixed point), the dynamic equations $\dot{x} = f(x)$ will take you immediate away from points at which $dV/dt = 0$.

It turns out that this method is easier to use for asymptotic stability than constructing a Lyapunov function which is strictly less than zero.

Lyapunov unstable

A fixed point x^* is **Lyapunov unstable** if we can find a Lyapunov function V, such that

- $V(x) = 0$ if and only is $x = x^*$ is the fixed point.
- $V(x) > 0$ in a neighborhood of x^*, when $x \neq x^*$.
- $dV/dt > 0$ in the neighborhood of x^*, when $x \neq x^*$.

As before, if we can only show $dV/dt \geq 0$, we can again invoke the LaSalle's Invariance Principle, showing that $dV/dt = 0$ cannot persist for any extended duration if it occurs.

Further, it turns out that we can weaken the above sufficient conditions for Lyapunov instability even further. Informally, it is sufficient that $V(x) > 0$ at some (not all) points that are arbitrarily close to x^*, not necessarily in a neighborhood of x^*. (see Richard Rand's lecture notes).