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Chaos in a soda can: Non-periodic rocking of upright cylinders with sensitive dependence on initial conditions

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ABSTRACT

It has been known for over a hundred years that all physically realizable motions of an ideal axisymmetric cylinder, rolling without slip on its flat circular bottom atop a flat horizontal surface, are periodic in the cylinder's angular rates and quasi-periodic overall. Here, we show that slight asymmetries in the cylinder, say the addition of an off-center point-mass, result in non-periodic motions with exponentially increasing sensitivity to initial conditions, making the dynamics chaotic.

1. Introduction

One of the most common table-top experiments that one performs unwittingly, say when sitting at a table after dinner, is casual play with objects found on the table. Most of us have some experience fiddling with empty soda cans, beer bottles, or other cylindrical containers on the table, including spinning them, letting them rock on their circular bottoms, or rolling them on their sides. However, all these motions come to an end quite quickly due to various frictional effects. As a consequence, one does not really have direct experience of the motions of cylinders in the absence of dissipation – motions, that once begun, last for ever. For over a hundred years (e.g., Chaplygin, 1948; Appell, 1900; Korteweg, 1900), it has been known that all the motions of ideal disks and cylinders rolling without slip on the edges of their flat circular bottoms are generally quasi-periodic, with the angular rates being exactly periodic. See O'Reilly (1996), Borisov and Mamaev (2002), for instance, for reviews of the principal results and the relevant literature.

Here, we show that breaking the axisymmetry of the cylinder with an off-center point-mass makes the angular rates non-periodic. Further, it appears that these non-periodic motions depend sensitively on initial conditions, with the sensitivity increasing exponentially in time, suggesting that the simplest table-top experiment of rocking asymmetric cylinders, albeit in the absence of dissipation, exhibits deterministic chaos.

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2. Model

We consider an axisymmetric cylinder with mass m, with a flat circular bottom of radius R, with the center of mass at a height H from the bottom when the cylinder is upright. The position of the center of mass G of the cylinder is, in general, at (x_G, y_G, z_G) with respect to an inertial frame with coordinate axes $\mathbf{e}_x$–$\mathbf{e}_y$–$\mathbf{e}_z$. The orientation of the cylinder is characterized by Euler angles, ψ, ϕ, and θ, as shown in Fig. 1, defining the following three rotations in sequence. The reference orientation of the cylinder is vertical in the inertial frame. The first rotation is by an angle ψ about the $\mathbf{e}_z$ axis, giving the coordinate frame $\mathbf{e}_{x1}$–$\mathbf{e}_{y1}$–$\mathbf{e}_{z1}$. The second rotation is by an angle ϕ about the negative $\mathbf{e}_{y1}$ axis, giving the coordinate frame $\mathbf{e}_{x2}$–$\mathbf{e}_{y2}$–$\mathbf{e}_{z2}$. The last rotation is by an angle θ about the symmetry axis $\mathbf{e}_{z2}$, giving the body-fixed frame $\mathbf{e}_{x3}$–$\mathbf{e}_{y3}$–$\mathbf{e}_{z3}$. The moment of inertia of the cylinder about the symmetry axis $\mathbf{e}_{z3}$ is C and the moment of inertia about any axis perpendicular to $\mathbf{e}_{z3}$ and passing through the center of mass is A. The acceleration due to gravity is g. The notation introduced so far is identical to that used in Srinivasan and Ruina (2008). The axisymmetry of the cylinder is broken by a point-mass m_o attached rigidly to the cylinder at point E, at the height of the center of mass, such that $\mathbf{r}_{GE} = R\mathbf{e}_{x3}$, as shown in Fig. 1c.

Using angular momentum balance about the center of mass G and linear momentum balance to obtain the contact forces, we obtain three second order ODEs that determine the evolution of the orientation ψ, ϕ, and θ. See Srinivasan and Ruina (2008) for a derivation of these equations in the absence of the additional point-mass m_o. With m_o, there are a few extra terms in the equations. We used MATLAB's symbolic manipulation capabilities to obtain these equations. The equations are given in the appendix. These equations are integrated using an adaptive stiff integrator (MATLAB's ode15s). The quality of the integration and indeed, the correctness of the equations are buttressed by the observation that the total energy was a constant in the numerical integration, up to an error of only about 10^{-8}, the integration tolerance.

3. Results and discussion

In the following, we use a cylinder with $m = 1$, $R = 5.1 \times 10^{-2}$, $H = 6.9 \times 10^{-2}$, $A = 5.13 \times 10^{-3}$, $C = 8 \times 10^{-3}$, and $g = 9.8$, all in consistent units. The phenomena described here are robust to large changes in these parameter values.

Fig. 2 shows the results of integrating these equations for an axisymmetric cylinder ($m_o = 0$). The angular rates $\dot{\phi}, \dot{\psi}, \dot{\theta}$ are all periodic, as is the angle ϕ. The angles ψ and θ are monotonic, in this case (not shown, but see Srinivasan and Ruina, 2008). On the other hand, Fig. 3 shows the results of integrating these equations for an asymmetric cylinder ($m_o = 0.3$). We see that the motion appears non-periodic and seems to have no simple pattern (although this does not rule out quasi-periodicity).

A key difference between the symmetric and asymmetric cases shown in Figs. 2 and 3 is how a small error in initial conditions grows with time. For both cases, we first compute the Jacobian $J(t)$ of the state of the system $w(t) = [\psi(t)\ \dot{\psi}(t)\ \phi(t)\ \dot{\phi}(t)\ \theta(t)\ \dot{\theta}(t)]^T$ with respect to the initial state $w(0) = [\psi(0)\ \dot{\psi}(0)\ \phi(0)\ \dot{\phi}(0)\ \theta(0)\ \dot{\theta}(0)]^T$. This Jacobian was computed by integrating the appropriate adjoint equations (obtained by symbolic differentiation of the equations of motion with respect to the states), rather than using finite differences, making the Jacobian estimate as accurate as the state variables themselves. Using adjoint equations also avoids saturation effects that arise when finite differences are applied to bounded state variables (Kantz and Schreiber, 1997).

The singular values of the Jacobian quantify the size of the ellipsoid that a sufficiently small ball of initial conditions gets stretched into. The logarithm of the maximum singular value of the Jacobian $\mu(t)$ is plotted in Fig. 4, showing that μ increases linearly with t for the axisymmetric case and increases exponentially with t for the asymmetric case. Note that the maximal Lyapunov exponent λ is given by $\lim_{t\to\infty} \log \mu(t)/t$, and appears to be zero for the axisymmetric case and about 0.3 for the

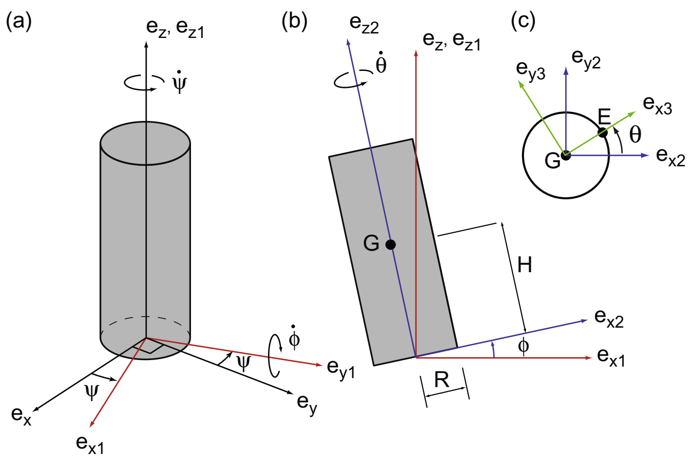

Fig. 1. Definition of the Euler angles and coordinate axes used to define the orientation of the container. Adapted from Srinivasan and Ruina (2008).

asymmetric case (extrapolating the fits). This suggests that the dynamics of the asymmetric cylinder are chaotic, with exponential sensitivity to initial conditions. A linear-with-t increase in the sensitivity to initial conditions is not considered chaos.

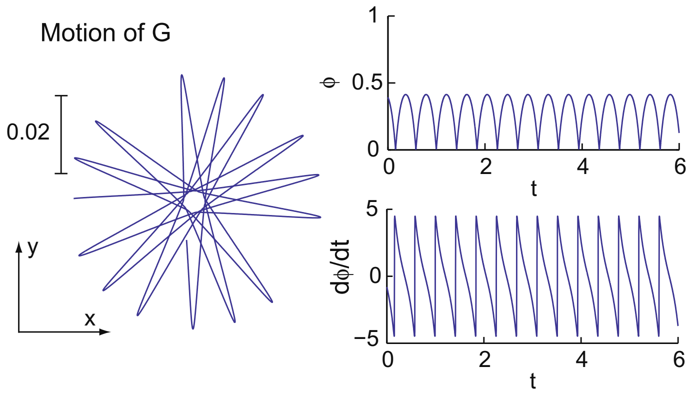

Fig. 2. Motion of an axisymmetric cylinder, with $m_o = 0$. Initial condition $\psi(0) = 0, \dot{\psi} = 0.3, \phi(0) = \pi/8, \dot{\phi}(0) = -0.8, \theta(0) = \pi, \dot{\theta}(0) = 0$. The state variables ϕ and $\dot{\phi}$ are periodic. The angular rates $\dot{\psi}$ and $\dot{\theta}$ are also periodic, but the angles ψ and θ need not be periodic, but can be monotonic increasing or decreasing (not shown). The x–y motion of the center of mass G is quasi-periodic in general, in that the trajectory may never close on itself.

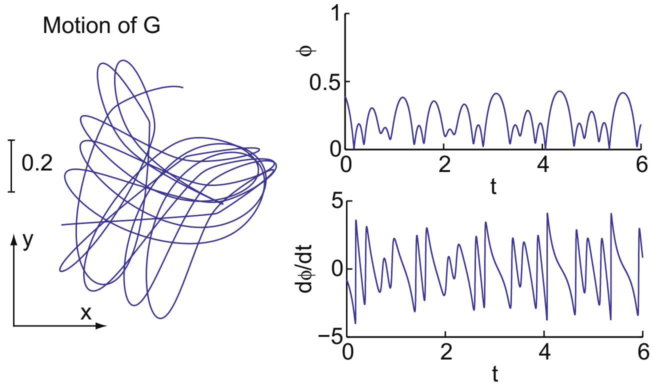

Fig. 3. Motion of an asymmetric cylinder, with $m_o = 0.3$. Initial condition $\psi(0) = 0, \dot{\psi} = 0.3, \phi(0) = \pi/8, \dot{\phi}(0) = -0.8, \theta(0) = \pi, \dot{\theta}(0) = 0$. The x–y motion of the point G (center of mass of the axisymmetric cylinder, without the off-center mass) shows no simple pattern and ϕ superficially appears to be non-periodic.

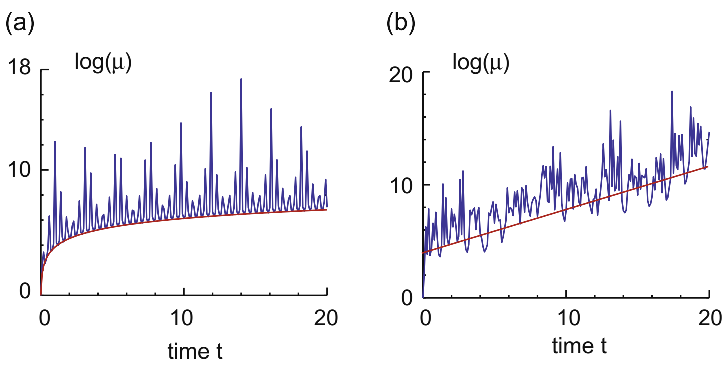

Fig. 4. Logarithm of the maximum singular value $\mu(t)$ of the Jacobian $J(t)$ is plotted as a function of time t, for (a) the axisymmetric case, and (b) the asymmetric case. In panel (a), the lower envelope of $\log\mu$ is well-approximated by $\log(1 + 46t)$, indicating that $\mu(t)$ scales linearly with t. In panel (b), we see that $\log\mu$ scales linearly with t; that is, μ scales exponentially with t.

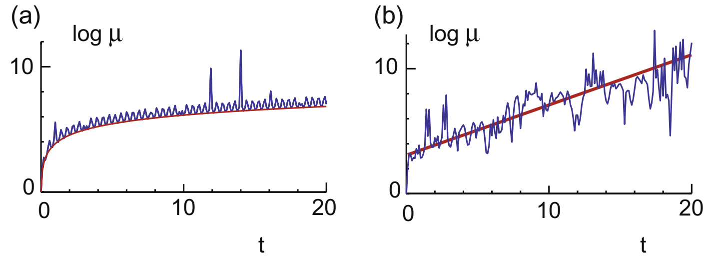

Fig. 5. Logarithm of the maximum singular value $\mu(t)$ of the reduced non-square Jacobian $J_B(t)$ is plotted as a function of time t, for (a) the axisymmetric case, and (b) the asymmetric case. In panel (a), the lower envelope of $\log\mu$ is well-approximated by $\log(1 + 46t)$, indicating that $\mu(t)$ scales linearly with t. In panel (b), we see that $\log\mu(t)$ scales linearly with t; that is, μ scales exponentially with t.

For the axisymmetric cylinder, the linear increase in sensitivity of the state to initial conditions is due to the fact that the period of the periodic motion changes (generically) with a small change in initial condition.

The large spikes in $\mu(t)$ are related to the singularity in the equations of motion when $\phi \approx 0$, when the contact point has a tendency to move extremely rapidly, accompanied by rapid and almost instantaneous changes in the angles ψ and θ. See Srinivasan and Ruina (2008) for a discussion of this singularity and its physical implications.

Exponential separation of trajectories is interesting, but is dubbed chaos only when the state variables themselves remain bounded, or at least do not grow to infinity exponentially in time. For our cylinder, conservation of energy implies that $\dot{\phi}$ and the linear velocity of any material point on the cylinder is bounded. If the motions are such that the cylinder always remains over ground, then the angle ϕ is also bounded. Bounded linear velocities imply that the position of any material point on the cylinder can go to infinity at most linearly in time t. But the same cannot be argued of the other variables. In particular, the angular rates $\dot{\psi}$ and $\dot{\theta}$ can potentially grow very large close to the singularity $\phi = 0$ (again, see Srinivasan and Ruina, 2008). Therefore, we repeated the calculation of $\mu(t)$ with a non-square block J_B of the full Jacobian J, measuring the sensitivity of the necessarily bounded outputs $[\phi(t)\ \dot{\phi}(t)]^T$ to the full initial state $w(0) = [\psi(0)\ \dot{\psi}(0)\ \phi(0)\ \dot{\phi}(0)\ \theta(0)\ \dot{\theta}(0)]^T$. The plot of $\log\mu(t)$ for this reduced Jacobian is shown in Fig. 5, indicating that the sensitivity of the bounded variables ϕ and $\dot{\phi}$ also grow, respectively, linearly and exponentially for the axisymmetric and asymmetric cases.

Not all initial conditions result in a chaotic trajectory for the asymmetric cylinder. Fig. 6 shows the results of a simulation for an asymmetric cylinder with $m_o = 0.05$. The plot of ϕ shows that the motion is apparently non-periodic, but we find by plotting $\log\mu(t)$ that the sensitivity to initial conditions only grows linearly with time t.

So far in this paper, we only considered motions in which the cylinder stays relatively close to upright for all time; for instance, ϕ does not grow beyond $\pi/2$, resulting in a toppled cylinder. Such toppling was avoided by choosing the initial condition with sufficiently low total energy E, and small tilt ϕ. It can be shown that the sufficient conditions for avoiding toppling (for generic initial conditions) are the satisfaction of $E < E^*$ and $\phi < \phi^*$, where $\phi^* = \tan^{-1}(pR/H)$ and $E^* = mg\sqrt{H^2 + p^2R^2}$ with $p = m/(m + m_o)$, using the convention that the potential energy is zero when the center of mass coincides with the ground. In our simulations, violating these conditions always led to the cylinder toppling, with ϕ reaching $\pi/2$. Because the equations of motion do not include a condition that detects the collision of the cylinder's side with the ground, the cylinder continues to move beneath the ground. In limited numerical experiments, we found that these motions always had sensitive dependence on initial conditions when the cylinder was asymmetric.

There are likely many-parameter families of these chaotic trajectories. In particular, because energy is conserved, a generic perturbation of the initial condition results in a trajectory with different energy, and therefore cannot ever converge to the unperturbed trajectory.

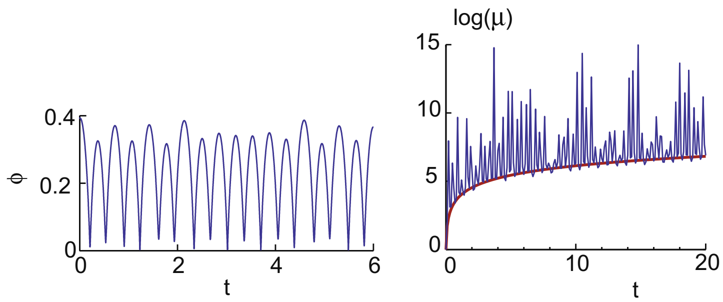

Fig. 6. Results for an asymmetric cylinder with $m_o = 0.05$, starting from initial conditions $\psi(0) = 0, \dot{\psi}(0) = 0.6, \phi(0) = \pi/8, \dot{\phi}(0) = 0.1, \theta(0) = \pi$, and $\dot{\theta}(0) = 0$. Logarithm of the maximum singular value $\mu(t)$ of the full Jacobian $J(t)$ is plotted as a function of time t and is well-approximated by $\log(1 + 46t)$, indicating that $\mu(t)$ scales linearly with t.

The system considered here is non-Hamiltonian, being conservative but nonholonomic (Neimark and Fufaev, 1967; Ruina, 1998). While there has been a long tradition of Hamiltonian chaos in the dynamical systems literature (e.g., Guckenheimer and Holmes, 1983; Wiggins, 1990), analogous results and methods for conservative nonholonomic systems appear hard to find. Perhaps one might conjecture that many other simple conservative nonholonomic mechanical systems, say an idealized bicycle (e.g., Meijaard et al., 2007) will also exhibit chaos when the mass distribution is asymmetric. Preliminary results suggest that an asymmetric cylinder on a frictionless table also has chaotic trajectories (this is a Hamiltonian system). Chaotic motions for similar Hamiltonian systems such as the asymmetric top have been noted previously (Holmes and Marsden, 1983; Barrientos et al., 1995; van der Heijden and Thomson, 2002; Borisov et al., 2008).

If only one had some way of making cylinders rock for a long-enough duration on a table, perhaps Poincaré would not have had to look to the heavens for inspiration leading to his discovery of deterministic chaos.

Acknowledgement

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Appendix A. Equations of motion for a cylinder rolling without slip

The equations of motion for the cylinder rolling without slip reduce to the form $Q_{i1}\ddot{\psi} + Q_{i2}\ddot{\phi} + Q_{i3}\ddot{\theta} = S_i$, with $i = 1, 2, 3$, in which

$$\begin{aligned}
Q_{11} &= A\sin\phi - (m+m_o)HR\cos\phi + mH^2\sin\phi + m_o(R^2+H^2)\sin\phi \\
&\quad - m_oR^2\sin\phi\cos^2\theta - m_oHR\cos\phi\cos\theta, \\
Q_{12} &= m_oR^2\sin\theta(1+\cos\theta), \\
Q_{13} &= -(m+m_o+m_o\cos\theta)HR,\ \ Q_{21} = -m_oR\sin\theta(R\cos\theta\sin\phi + H\cos\phi + R\sin\phi) \\
Q_{22} &= -A - (m+m_o)(H^2+R^2) - m_oR^2\cos^2\theta - 2m_oR^2\cos\theta,\ \ Q_{23} = -m_oHR\sin\theta \\
Q_{31} &= C\cos\phi + mR^2\cos\phi - mRH\sin\phi + 2m_oR^2\cos\theta\cos\phi \\
&\quad - m_oRH\cos\theta\sin\phi + 2m_oR^2\cos\phi - m_oRH\sin\phi \\
Q_{32} &= m_oHR\sin\theta,\ \ Q_{33} = C + 2m_oR^2\cos\theta + mR^2 + 2m_oR^2
\end{aligned} \tag{1}$$

$$\begin{aligned}
S_1 &= -2m_oR^2\dot{\theta}\dot{\psi}\sin\theta\sin\phi\cos\theta + 2m_oR^2\dot{\theta}\dot{\phi}\sin^2\theta - m_oR^2\dot{\psi}\dot{\theta}\sin\phi\sin\theta \\
&\quad - m_oR^2\dot{\psi}^2\sin\phi\cos\phi\sin\theta\cos\theta - m_oR^2\dot{\psi}^2\sin\theta\sin\phi\cos\phi - m_oRH\dot{\psi}^2\sin\theta\cos^2\phi \\
&\quad + m_oRH\dot{\phi}^2\sin\theta - 2mH^2\dot{\psi}\dot{\phi}\cos\phi - 2A\dot{\psi}\dot{\phi}\cos\phi + C\dot{\phi}\dot{\psi}\cos\phi \\
&\quad - m_ogR\sin\theta\cos\phi + C\dot{\phi}\dot{\theta} - 2m_oH^2\dot{\psi}\dot{\phi}\cos\phi - 2mHR\dot{\psi}\dot{\phi}\sin\phi \\
&\quad - 2m_oHR\dot{\phi}\dot{\psi}\sin\phi\cos\theta - 2m_oHR\dot{\psi}\dot{\phi}\sin\phi - m_oRH\dot{\theta}^2\sin\theta - 2m_oRH\dot{\theta}\dot{\psi}\sin\theta\cos\phi
\end{aligned} \tag{2}$$

$$\begin{aligned}
S_2 &= -m_oR\dot{\psi}^2H(1+\cos\theta) + C\dot{\psi}^2\sin\phi\cos\phi + mR^2\dot{\theta}\dot{\psi}\sin\phi + mg(R\cos\phi - H\sin\phi) \\
&\quad + m_oR^2\dot{\psi}^2\cos^2\theta\sin\phi\cos\phi + m_oR^2\dot{\psi}^2\sin\phi\cos\phi - m_ogH\sin\phi - mRH\dot{\psi}^2 \\
&\quad + m_oHR\dot{\theta}\dot{\psi}\cos\phi + C\dot{\psi}\dot{\theta}\sin\phi + 2mHR\dot{\psi}^2\cos^2\phi + m_oRH\dot{\theta}^2\cos\theta \\
&\quad - m_oH^2\dot{\psi}^2\cos\phi\sin\phi + mHR\dot{\psi}\dot{\theta}\cos\phi + m_ogR\cos\phi + m_ogR\cos\theta\cos\phi - A\dot{\psi}^2\cos\phi\sin\phi \\
&\quad + 2m_oHR\dot{\psi}^2\cos^2\phi + 2m_oR^2\dot{\psi}^2\cos\theta\sin\phi\cos\phi + 2m_oRH\dot{\psi}^2\cos\theta\cos^2\phi \\
&\quad - 2m_oR^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta + 3m_oR^2\dot{\psi}\dot{\theta}\cos\theta\sin\phi + 2m_oRH\dot{\theta}\dot{\psi}\cos\theta\cos\phi - mH^2\dot{\psi}^2\cos\phi\sin\phi \\
&\quad - 2m_oR^2\dot{\theta}\dot{\phi}\sin\theta + 2m_oR^2\dot{\theta}\dot{\psi}\cos^2\theta\sin\phi + mR^2\dot{\psi}^2\sin\phi\cos\phi + m_oR^2\dot{\psi}\dot{\theta}\sin\phi
\end{aligned} \tag{3}$$

$$\begin{aligned}
S_3 &= C\dot{\phi}\dot{\psi}\sin\phi + 2m_oR^2\dot{\phi}\dot{\psi}\cos^2\theta\sin\phi + 4m_oR^2\dot{\psi}\dot{\phi}\cos\theta\sin\phi + 2m_oRH\dot{\psi}\dot{\phi}\cos\theta\cos\phi \\
&\quad + m_oR^2\dot{\psi}^2\cos\theta\sin\theta - m_oR^2\dot{\phi}^2\sin\theta\cos\theta + m_oR^2\dot{\theta}\dot{\psi}\sin\theta\cos\phi - m_oR^2\dot{\psi}^2\sin\theta\cos^2\phi\cos\theta \\
&\quad - m_oR^2\dot{\psi}^2\sin\theta\cos^2\phi + m_oRH\dot{\psi}^2\sin\theta\cos\phi\sin\phi + 2mR^2\dot{\psi}\dot{\phi}\sin\phi + 2m_oR^2\dot{\psi}\dot{\phi}\sin\phi \\
&\quad + 2mRH\dot{\phi}\dot{\psi}\cos\phi + 2m_oHR\dot{\psi}\dot{\phi}\cos\phi + m_ogR\sin\theta\sin\phi - m_oR^2\dot{\phi}^2\sin\theta \\
&\quad + m_oR^2\dot{\theta}^2\sin\theta + m_oR^2\dot{\psi}^2\sin\theta
\end{aligned} \tag{4}$$

References

Appell, P., 1900. Sur l'intégration des équations du mouvement d'un corps pesant de révolution roulant par une arête circulaire sur up plan horizontal; cas parficulier du cerceau. Rendiconti del Circolo Matematico di Palermo 14, 1–6.

Barrientos, M., Perez, A., Ranada, A.F., 1995. Weak chaos in the asymmetric heavy top. European Journal of Physics 16, 106–112.

Borisov, A.V., Kilin, A.A., Mamaev, I.S., 2008. Chaos in a restricted problem of rotation of a rigid body with a fixed point. Regular and Chaotic Dynamics 13 (3), 221–233.
Borisov, A.V., Mamaev, I.S., 2002. The rolling motion of a rigid body on a plane and a sphere. Hierarchy of dynamics. Regular and Chaotic Dynamics 7, 177–200.
Chaplygin, S.A., 1948. On motion of heavy rigid body of revolution on horizontal plane. In: M.-L. (Eds.), Collection of Works, vol. 1, OGIZ, 1948, pp. 57–75.
Guckenheimer, J., Holmes, P., 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer.
Holmes, P.J., Marsden, J.E., 1983. Horseshoes and Arnold diffusion for Hamiltonian systems on lie groups. Indiana University Mathematics Journal 32 (2), 273–309.
Kantz, H., Schreiber, T., 1997. Nonlinear Time Series Analysis. Cambridge University Press, New York.
Korteweg, D., 1900. Extrait d'une lettre à M. Appel. Rendiconti del Circolo Matematico di Palermo 14, 78.
Meijaard, J.P., Papadopoulos, J.M., Ruina, A., Schwab, A.L., 2007. Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review. Proceedings of the Royal Society Series A 463, 1955–1982.
Neimark, Ju.I., Fufaev, N.A., 1967. Dynamics of Nonholonomic Systems. American Mathematical Society.
O'Reilly, O.M., 1996. The dynamics of rolling disks and sliding disks. Nonlinear Dynamics 10, 287–305.
Ruina, A., 1998. Non-holonomic stability aspects of piecewise-holonomic systems. Reports on Mathematical Physics 42, 91–100.
Srinivasan, M., Ruina, A., 2008. Rocking and rolling: a can which appears to rock might actually roll. Physical Review E 78, 066609.
van der Heijden, G.H.M., Thomson, J.M.T., 2002. The chaotic instability of a slowly spinning asymmetric top. Mathematical and Computer Modeling 36, 359–369.
Wiggins, S., 1990. Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer.