Abstract

Jumping on trampolines is a popular backyard recreation. In some of these trampoline games (e.g., “seat drop war”), when two people land on the trampoline with only a small time-lag, one person bounces much higher than the other, as if energy has been transferred from one to the other. First, we illustrate this energy-transfer in a table-top demonstration, consisting of two balls dropped onto a mini-trampoline, landing almost simultaneously, sometimes resulting in one ball bouncing much higher than the other. Next, using a simple mathematical model of two masses bouncing passively on a massless trampoline with no dissipation, we show that with specific landing conditions, it is possible to transfer all the kinetic energy of one mass to the other through the trampoline. For human-like parameters, starting with equal energy, the energy transfer is maximal when one person lands approximately when the other is at the bottom of her bounce. The energy transfer persists even for very stiff surfaces. The mathematical model exhibits complex non-periodic long-term motions, with the energy being shuttled from one mass to the other ad infinitum. As a complement to this passive bouncing model, we also performed a game-theoretic analysis, considering a zero-sum game in which each player’s goal is to gain the other player’s kinetic energy during a single passive bounce, by possibly extending her leg during flight. We find, for high enough initial energy and an otherwise symmetric situation, that the best strategy for both subjects (minimax strategy and Nash equilibrium) is to use the shortest available leg length and not extend their legs. On the other hand, an initial asymmetry in energies allows the player with more energy to gain even more energy in the next bounce, thus making symmetric bouncing unstable in the game-theoretic sense of mixed Nash equilibria and minimax strategies.

Introduction

Bouncing on a trampoline has evolved from a backyard activity for children to an Olympic Sport. While Olympic trampolining only has one person bouncing on a trampoline, in its recreational form, it is quite common for more than one person to bounce on the trampoline simultaneously. In particular, children play a two-person game on trampolines called “seat drop war.” In this game, each player bounces alternatively with her feet and her ‘seat’ (being in an L-shaped body configuration), as shown in Figure 1. See also Supplementary Information video, showing this game being played. Each player is able to increase her mechanical energy while bouncing (jumping) with her feet by performing mechanical work with her legs, but she essentially bounces passively when bouncing on her seat (and probably loses some energy due to damping). The goal of this game is to be the last player bouncing. As the game progresses with the two players bouncing alternatively with their feet and their seat, the relative phasing between their bounces typically changes: sometimes the players bounce out of phase and sometimes they bounce in phase. The game often ends with one person having so little upward velocity when bouncing on her seat that she is unable to get back on her feet for the next bounce. Often, associated with this loss, the second player appears to have gained most of the energy lost by the first player, thereby bouncing higher than usual. This article is motivated by this apparently catastrophic energy transfer between the players, which typically happens during a bounce in which the two players are simultaneously in contact.
with trampoline for some overlapping time period.

Here, we show that the dramatic energy transfer is observed even in the passive bouncing of inanimate masses. We first describe a simple physical demonstration of the energy transfer: dropping two balls simultaneously onto a small trampoline sometimes results in one ball bouncing much higher than the other. Then, we construct a simple energy-conservative mathematical model, with the two people modeled as masses bouncing passively on a trampoline. This model also exhibits the dramatic energy transfer observed in seat drop war. We show that there is typically an optimal difference between the landing times of the two masses (hereafter called the ‘contact time-lag’) that maximizes energy transfer. The mathematical model, in absence of dissipation or sideways movement of masses, displays complex non-periodic motion, with repeated transfer of energy between the two masses.

Finally, we make a first step at analyzing the game, not as a simple passive dynamics problem involving two balls bouncing, but as a strategic competitive game between two players from a game theoretic perspective, obtaining the optimal strategies for the two players for the zero-sum game.

Results

A physical demonstration: Two balls on a trampoline

To illustrate that energy transfer between people on a trampoline can happen through purely passive mechanisms, we designed a simple table-top demonstration involving a store-bought mini-trampoline and two balls (see also Materials and Methods).

Figure 2 shows a series of key frames, illustrating the energy transfer between the two (tennis) balls, dropped nearly but not exactly simultaneously. The two balls contact the trampoline at slightly different times, with some overlapping period when they are both in contact with the trampoline. The mass that makes contact with the trampoline second bounces much higher. See also video in Supplementary Information, which shows this specific example in slow motion, and also other examples illustrating how when the masses make contact with the trampoline approximately simultaneously, they bounce up to about the same height.

We did not perform carefully controlled drops, make detailed measurements of the resulting bounces, or try to make this table-top experiment a dynamically scaled version of two humans bouncing on a larger trampoline. We intend this only as a demonstration of the phenomenon.

When dropped by human hands, the two balls often land at slightly different times due to human motor variability, resulting in different amounts of overlap between their contact phases with the trampoline. As a consequence, as seen from the mathematical models below, the rise heights of the masses after the bounce have corresponding variability. When there is no contact overlap, as happens often (if the drops are not nearly simultaneous), there is no dramatic energy transfer.
A mathematical model: passive bouncing of two masses

When people bounce on trampolines, they perform positive mechanical work with their legs to counteract any loss of energy (through passive dissipation or active negative leg work). Here, for simplicity, we restrict ourselves to energy-conservative models: no leg work or dissipation. See Materials and Methods for simulation details for these mathematical models.

We idealize the two players as particles with masses $m_1$ and $m_2$ bouncing on a trampoline, modeled as a taut massless string of length $L$, as shown in Figure 2a-b. The two masses are at horizontal distances $a$ and $c$ respectively from the nearest fixed ends of the trampoline; the distance between the masses is $b = L - (a + c)$. The trampoline is at a large initial tension $T$, so that the tension in it does not change to first order when deformed (Fig. 2b). The particles do not slip against the trampoline, and we neglect the horizontal forces on the particles, so that the motion of the particles is purely vertical, for all time.

The vertical position of the two masses are denoted $y_1$ and $y_2$, positive upward (Figure 2b). The undeflected trampoline is at $y = 0$. We divide the state space into four phases based on which masses are in contact with the trampoline: P0 (neither in contact with trampoline, both in flight), P1 (only $m_1$ in contact), P2 (only $m_2$ in contact), P12 (both in contact). See Figure 2a. The corresponding equations of motion are:

P0: $m_1\ddot{y}_1 = -m_1g$, $m_2\ddot{y}_2 = -m_2g$

P1: $m_1\ddot{y}_1 = -m_1g + T\left(\frac{y_1}{a} + \frac{y_1}{b+c}\right)$, $m_2\ddot{y}_2 = -m_2g$

P2: $m_1\ddot{y}_1 = -m_1g$, $m_2\ddot{y}_2 = -m_2g + T\left(\frac{y_2}{c} + \frac{y_2}{a+b}\right)$

P12: $m_1\ddot{y}_1 = -m_1g + T\left(\frac{y_1}{a} + \frac{y_1 - y_2}{b}\right)$,

$$m_2\ddot{y}_2 = -m_2g + T\left(\frac{y_2}{c} - \frac{y_1 - y_2}{b}\right).$$

These equations are linear, and assuming small vertical deflections of the trampoline. The total dynamical system, consisting of these four phases patched together, is piecewise linear, and therefore, ultimately, nonlinear and non-smooth. Transition between phases occurs with no discontinuous change in position and velocity. A mass leaves the trampoline when the upward force on it by the trampoline becomes
zero while it is moving upward. This take-off event coincides with the trampoline becoming a single straight line to the left and to the right of the mass; that is, the deflection in the trampoline vanishes locally. We assume two take-off or landing events do not happen simultaneously. The trampoline comes immediately to rest when neither mass is in contact; that is, the trampoline has no intrinsic dynamics. These constitutive assumptions are consistent with energy conservation.

The total system energy consists of the kinetic and gravitational potential energies of the two masses, namely \( (m_1v_1^2/2) \) and \( m_2gy \), respectively, and the stored energy in the stretched string. The stored energy in the string in the various phases are:

\[
P0: \quad E_{\text{string}} = 0, \quad (2)
\]

\[
P1: \quad E_{\text{string}} = \frac{T}{2} \left( \frac{1}{a} + \frac{1}{b + c} \right) y_1^2, \quad (3)
\]

\[
P2: \quad E_{\text{string}} = \frac{T}{2} \left( \frac{1}{c} + \frac{1}{a + b} \right) y_2^2, \quad (4)
\]

\[
P12: \quad E_{\text{string}} = \frac{T}{2} \left[ \frac{y_1^2}{a} + \frac{y_2^2}{c} + \frac{(y_1 - y_2)^2}{b} \right] \quad (5)
\]

In the following discussions, we will sometimes refer to the “energy of a particular mass,” implicitly partitioning the total system energy between the two masses. The partitioning of the total system mechanical energy into the two masses is clearest when both masses are in flight — then, each mass is associated with the sum of its kinetic and gravitational potential energy. When both masses are in contact with the trampoline, there is no objective partitioning of the total energy between the two masses. When exactly one mass is in contact with the trampoline, we use the convention that the mass that is in contact gets credit for the energy stored in the string.

See Materials and Methods for how this non-smooth dynamical system is simulated.
Figure 4. Long-term bouncing dynamics. a) The motion $y_1(t)$ and $y_2(t)$ starting at rest from initial conditions $y_1(0) = 1$ and $y_2(0) = 0.9$. b) The total energy (kinetic + gravitational potential) in mass-1 when both masses are in flight, as a fraction of the total energy. c) The state of the system when the right mass either just takes off (red dots) or when the right mass just lands (blue dots), when the left mass is already in the air.

Passive dynamics predicts energy transfer and complex dynamics

For the following simulations of the above model pertaining to bouncing people (as opposed to bouncing tennis balls described later), we use the following parameters: $m_1 = m_2 = 70$ kg, $L = 3.5$ m, $a = b = c = L/3$, and $g = 9.81$ ms$^{-2}$. The vertical stiffness of the trampoline at its midpoint $L/2$ is $4T/L$. We picked tension $T$ such that this midpoint stiffness was equal to 5000 N/m, roughly the secant stiffness of the trampoline described in [1].

Before considering a single bounce and the energy transfer in greater detail, we examine simulations of the passive dynamical system for a long time period. Simulating this dynamical system from any generic initial condition (which does not result immediately in the two masses touching the trampoline simultaneously in the first bounce results) in a complex non-periodic bouncing motion of the two masses, as shown in Figure 4a. Also, see video of the animation in the Supplementary Information.

The two masses repeatedly exchange energy with each other, sometimes one mass bounces higher and sometimes the other mass bounces higher: Figure 4b shows the fluctuating energy content in mass-1. Thus, over a long enough simulation, if the “game” is stopped at some random moment sufficiently far into the future, there is equal likelihood of either player “winning” i.e., having more energy. The energy transfer between masses occurs when the masses are in simultaneous contact with the trampoline, performing work on each other through the trampoline. Thus, it appears that the passive model is sufficient to explain the energy transfer.

To be clear, while the mechanics of a single bounce interaction of the two masses may be comparable to that of the interaction between humans on a trampoline, the details of the long-time simulation may not be of direct applicability to long-time human bouncing. We discuss this long-time simulation further for its own intrinsic dynamical properties.

In a single long simulation, the state of the system appears to come arbitrarily close to almost every region of the accessible phase space, consistent with energy conservation. Figure 4c gives a scatter-plot of the snap-shots of the state, in a single long simulation lasting about $5 \times 10^4$ phases, at transitions between phases P0 and P2: when mass-2 lands ($y_2 = 0, v_2 > 0$, red dots) or takes off ($y_2 = 0, v_2 < 0$, blue dots), with mass-1 already in the air ($y_1 < 0$). The interior of the disk in Figure 4c is the set of all possible states consistent with constant total energy. We found that the histogram of energies for each mass over time is not a constant function of energy, but has a minimum near symmetric bouncing when each mass has half the energy. Indeed the thin slivers of empty regions in Figure 4c near $|v_2| \approx 4.201$ corresponds
to states at which each mass has exactly half the energy – in particular, \( v_2 = \sqrt{2gH_{\text{avg}}} = 4.201 \) where \( H_{\text{avg}} = 0.9 \) is the average height of the masses at initial condition. (We do not know if the system is ergodic [2].)

The complex dynamics observed for this dynamical system is not entirely unanticipated. A well-studied dynamical system is a mass bouncing, elastically or inelastically, on a much more massive paddle oscillating vertically and exactly sinusoidally [3,4]; this system is known to be chaotic in certain parameter regimes. Note that this mass on an oscillating paddle system can be obtained as a distinguished limit of our dynamical system by making \( m_1 \gg m_2 \) and by ensuring that \( m_1 \) never has a flight phase, so that it oscillates exactly sinusoidally. More recently, apparently unaware of this earlier work, similar chaotic dynamics were observed and analyzed for water droplets bouncing on a fluid surface, acting as a trampoline [5,6]. One qualitative difference between these earlier systems and the two mass system consider in this article, is that typically the paddle or the fluid trampoline in these systems is oscillated using external energy input, so that the total system energy need not be constant.

### Maximizing energy transfer

In this section (and in the Appendix), the passive bouncing model will be used to evaluate the effectiveness of various variables that players could control in order to gain energy from their opponent and win the “seat drop war” game. Players can control their jump timing relative to their opponent, their energy at impact, and the distance between themselves and the opponent. We find that players should aim to contact the trampoline when their opponent has maximally deflected the trampoline (half of the contact time), and should attempt to have more energy than the opponent (either due to larger mass or higher jump). The transfer of energy between players is larger when the players are closer together, than when they are farther apart.

Energy can get transferred only when both masses are in simultaneous contact with the trampoline. Without loss of generality, consider the situation in which mass-2 lands on the trampoline when mass-1 is already in contact, so that the two masses are in simultaneous contact with the trampoline for a while. Now simulate the system forward in time until both masses are in flight again i.e., phase P0 is reached. We examine the energy increase in the two masses when P0 is reached \(^1\), as a function of the time difference between when mass-1 makes contact and mass-2 makes contact with the trampoline — the contact time-lag (Figures 3a-c). The energy increase in the two balls is normalized by the pre-contact energy of each mass \( (\Delta E_i/E_i(0)) \), and the contact time-lag has been normalized by the contact period of mass-1 in the absence of interference by mass-2. The model parameters are as noted in the previous sub-section, including \( b = L/3 \), except when specifically overridden below.

The energy increase in Figure 3a-c are discontinuous functions of the contact time-lag. This discontinuity arises because we record the energy transfer only at the first transition to phase P0 after mass-2 lands. The number of phases that the system goes through before reaching phase P0 can depend on the initial conditions. Thus, on one side of a discontinuity, a mass barely takes off, with close to zero velocity. And on the other side of the discontinuity, this same mass comes very close to taking off, but does not have enough energy to do so, resulting in the system passing through more contact phases phases, P1, P2, and P12, before phase P0 can happen.

In Figure 3a, the masses had initially the same energy. So the normalized energy increases in the two masses are mirror images of each other about the x-axis because of energy conservation \( (\Delta E_1 = -\Delta E_2) \), and because they have been normalized by the same quantity \( (E_1(0) = E_2(0)) \).

\(^1\)Note that we have chosen to plot the energy in the two balls at the first moment when both balls are in flight phase. An alternative version of the plot would record the energies at the first moment one of the balls begins flight. In this alternate version, the ball still in contact will get credit for the stored elastic energy in the string. We note that this version of the plot (not shown) looks slightly different from the plots shown, as the ball in flight has the opportunity to re-contact the trampoline before the other ball takes off, providing further opportunity for energy exchange.
Figure 5. **Energy transfer and contact time-lag.** Energy increase in mass-1 (red) and mass-2 (blue) as a function of the impact time-lag. The masses are equal and are placed symmetrically on the trampoline, so that \( a = c = (L - b)/2 \). The separation between the masses \( b = L/3 \). a) The initial energies \( E_1(0) = E_2(0) \) are equivalent to dropping from rest from a height of 1 m. b) Mass-1’s initial energy \( E_1(0) \) is the same as for panel-a, but \( E_2(0) = 2E_2(0) \). c) Mass-1’s initial energy \( E_1(0) \) is the same as for panel-a, except mass-2’s energy is \( E_2(0) = E_1(0)/2 \).

Here, the mass that lands second, namely mass-2, always gains energy and mass-1 always loses energy, whatever the contact time-lag (Figure 3a). As would be expected, we see in Figure 3a that the energy transfer is close to zero when mass-2 lands when the mass-1 has just landed (close to symmetric simultaneous bounce), or about to take off (close to no interaction). More significantly, we see that the normalized energy increase of mass-2 reaches unity \( (\Delta E_2/E_1(0) = 1) \), around when the time-lag is about half of mass-1’s contact period. That is, if mass-2 makes contact when mass-1 is approximately half-way through its bounce, the energy transfer is essentially 100% in a single bounce.

When complete energy transfer occurs, mass-2 makes contact when mass-1 just starts to rise or just before it starts to rise (note the two contact time-lags on either side of contact timing = 0.5 for which the energy transfer is perfect). As mass-2 pulls the string down, mass-1 remains in contact for a brief while and then leaves contact with an upward velocity, earlier than it would otherwise have in the absence of mass-2. This mass-1’s upward velocity \( v_1 \) is such that, if left alone, the mass-1 would just about reach \( y = 0 \) at roughly the same time that mass-2 leaves contact, so that both masses are in flight with mass-1 with close to zero energy.

The basic energy transfer mechanism can be most simply understood using an ‘instantaneous argu-
ment' as illustrated in Figure 4. Given a state as in Figure 4a, in which mass-1 is rising while in contact with the string, the presence of mass-2 lowers the upwards vertical force on mass-1 (Fig 4c) compared to when mass-2 is not in contact (Fig 4b, keeping position of mass-1 fixed). Thus, for any given upward \( v_1 \), the corresponding acceleration \( v_1 \) and the instantaneous positive power on mass-1 by the string would be lower than if mass-2 were not in contact. If this situation persists until mass-1 takes off, it would take off with an upward speed smaller than if mass-2 had not interfered. Note that the situation described in this ‘instantaneous’ argument need not persist in general until phase P0 is reached — the bounce dynamics can be quite complex in certain parameter regimes. This heuristic ‘instantaneous’ argument is likely most directly applicable when mass-2 lands on the string when mass-1 is just about to take off, when it is likely that the situation presented in Fig 4a would persist until at least mass-1 takes off.

The details of the energy transfer’s dependence on the contact time-lag can be complex and dependent on various other system parameters including differences in energy of mass-1 and mass-2 before contact (Figure 3b-c). When mass-2 has twice mass-1’s initial energy (Figures 3b), mass-1 gains half of mass-2’s energy when both contact simultaneously (contact time-lag = 0). In contrast, when mass-2 has half mass-1’s initial energy (Figures 3c), mass-2 gains half of mass-1’s initial energy when they contact simultaneously (contact time-lag = 0).

Even though the motion is governed by relatively simple linear differential equations in each phase, an analytical treatment to obtain the dependencies in Figure 3 was found to be cumbersome (although likely feasible) because the differential equations had to be integrated until a certain event happened (namely one or both masses taking off) and patched together, rather than simply integrated until a particular time.

See the Appendix for further discussion of energy transfer between the balls under some simplifying limits, such as high initial energies and one mass much larger than the other.

### Stability of symmetric bouncing

When the masses are equal \( (m_1 = m_2) \) and symmetrically positioned \( (a = c) \), a symmetric periodic motion is achieved by dropping the two balls from the same height. There is a one-parameter family of such symmetric motions parameterized by the initial height. In simulations, we find that symmetric bouncing is unstable: a generic arbitrarily small perturbation of the initial conditions for such symmetric bouncing will lead to the two balls making contact not quite simultaneously, and this asymmetry grows with time.

The stability of symmetric motion can be analyzed more carefully by considering the properties of a ‘Poincare map’ [3, 7] as described below. Say the two balls are dropped from height \( H_0 \). We define the Poincare map as mapping states \( \{v_1, y_2, v_2\} \) on the Poincare section\(^2 \) \( y_1 = -H_1 \) (where \( |H_1| < |H_0| \) and \( y_1 > 0 \)) back to itself. We compute numerical (central difference) approximations to the Poincare map’s Jacobian. This Jacobian necessarily has one eigenvalue equal to 1, because of energy conservation. See Figure 5. For most initial heights \( H_0 \), the Jacobian has two non-unit real eigenvalues, with one real eigenvalue greater than one in absolute value, and the other real eigenvalue less than one. Thus symmetric bouncing displays linear instability for these heights. For a small range of heights, the two non-unit eigenvalues were complex conjugates with absolute value equal to one (within numerical error).

We further noticed that for all heights, the two non-unit eigenvalues, real or complex, were reciprocals of each other \( (\lambda \text{ and } 1/\lambda) \), a property shared by symplectic maps [8], and therefore, the product of all the eigenvalues equals one (accurate to about 4 decimal places); see footnote \(^3 \). Nevertheless, for all heights,

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\(^2\) A Poincare section is a surface in state space, transverse to the periodic orbit.

\(^3\) If a map is symplectic, all its eigenvalues being on the unit circle has implications for its stability. The usual definition of symplectic maps are for systems with even dimensions, hence does not apply to the Poincare map directly, which is three dimensional. However, the corresponding time-map — the mapping of the full set of states \( \{y_1, v_1, y_2, v_2\} \) over one time-period of symmetric bouncing is four-dimensional (and therefore even dimensional). Over its complete domain, this fully non-linear map would be non-smooth and we did not examine its symplecticity.
Figure 7. The eigenvalues of a return map on the periodic orbit corresponding to a symmetric bounce for symmetric masses. The product of the eigenvalues was equal to 1 (with an error of about $10^{-4}$). In the intermediate regime shown, all three eigenvalues, two of which are complex conjugates and reciprocals of each other, have unit absolute values.

even for heights $H_0$ where all eigenvalues seem to have unit magnitudes, long-enough simulations appear to eventually take the system far away from symmetric bouncing. For these simulations, we used a high accuracy integrator that preserved energy at a level of $10^{-7}$ over long durations of integration (but not a ‘symplectic’ integrator [9]).

Game-theoretic analysis: bouncing as a zero sum game

We began this article with a game played on the trampoline as motivation, but all the analysis so far has been of passive mechanical models. In this section, we consider a ‘game-theoretic’ analysis, not of the original game described in the introduction, but of a simplified abstraction. We only present a brief analysis. For the reader unfamiliar with game theory, we recommend [10] for a gentle non-technical introduction, and [11] for a more advanced mathematical treatment and precise definitions. For this section, we use parameter values pertaining to people, as used earlier ($m_1 = m_2 = 70$kg, etc).

Model with legs

Consider two players modeled as point-masses as shown in Figure 6a, now with mass-less legs that the players can extend and contract within a range of lengths. The leg lengths are respectively $L_1$ and $L_2$, and $0 \leq L_i \leq L_{\text{max}}$. The leg lengths are picked before contact, and during a bounce, the legs are completely rigid and perform no mechanical work. After interacting with the trampoline, eventually, both players reach flight phase (phase P0) and the leg lengths go back to zero. The object of the game, then, is for each player to pick their leg length $L_i$ so that their energy is maximum when they reach phase P0, when both are in flight again.

We analyze the energy transfer over only one bounce; that is, over only one complete interaction through the trampoline. Strategies for maximizing energy transfer over multiple bounces were not considered.

Picking a non-zero leg length has two possibly competing effects for a player: (1) It makes the player contact the trampoline earlier, thus altering the relative contact timing, perhaps giving the player an advantage. (2) It alters the initial ‘effective’ potential energy of the player.
A game: each player is allowed to pick an initial rigid leg length and then bounce passively.

Figure 8. Bouncing as a zero sum game. a) Two players drop from the same height and they can pick a rigid leg length to modify their contact times. b) The normalized energy increase in mass-1 (the payoff function) as a function of the leg length choices of the two players, when the two players drop from an initial height of $H = 2$ m. The vertical line (blue/red) shows the minimax and maximin strategy that coincide. c) The payoff function when initial height is $H = 0.9$ m. For this height, the pure minimax (denoted player-2) and maximin (denoted player-1) strategy do not coincide. The second panel shows the optimal probabilities corresponding to mixed minimax strategies.

### A zero-sum game

Player 1 wishes to maximize her energy increase $\Delta E_1$ and player 2 wishes to maximize her energy increase $\Delta E_2$. By energy conservation, we have $\Delta E_1 + \Delta E_2 = 0$, giving a zero-sum game, also called a strictly competitive game [11]. In this zero-sum game, player-1’s energy gain is player-2’s energy loss and vice versa. In other words, player 1 wishes to maximize $\Delta E_1$ and player 2 wishes to minimize $\Delta E_1$ by picking $L_1$ and $L_2$, respectively.

This is a ‘continuous game’ [12] in that the players pick continuous values variables $L_1$ and $L_2$.

However to compute the optimal strategies for the game, we discretize the continuous space of strategies, by using a $100 \times 100$ grid of $L_1 - L_2$ pairs, each ranging from 0 to $L_{\text{max}} = 0.5$ m. We compute the energy transfers for each $L_1 - L_2$ pair on this grid, by performing calculations similar to those that produced Figure 3. This gives a $100 \times 100$ energy transfer matrix $\Delta E_1$ (‘payoff matrix’ in usual game-theoretic terminology), a discrete approximation to energy transfer function $\Delta E_1(L_1, L_2)$ (‘payoff function’). This energy transfer function is shown as a surface in Figure 6b-c for two different cases in which the two masses have identical initial heights $H = 0.9$ m and 2 m respectively.

When the initial heights are equal and all other parameters are symmetric, we have a ‘symmetric’ zero-sum game [11], defined by $\Delta E_1(L_1, L_2) = \Delta E_2(L_2, L_1) = -\Delta E_1(L_2, L_1)$, or in terms of the payoff matrix, we have, $\Delta E_1 = -\Delta E_1^T$. The game is not symmetric if the players start with different heights or energies. Symmetry is relevant because it implies some properties of the game’s solution.

### Deterministic (pure) strategies

If the player 1 chooses her leg-length $L_1$ first and player 2 chooses $L_2$ second, then player 2 will pick the $L_2$ that minimizes the $\Delta E_1$ for the chosen $L_1$. Therefore, when player 1 chooses first, she would pick the $L_1$ for which the $\Delta E_1$ has the maximum minimum. That is, she performs the following optimization problem:

$$\max_{L_1} \min_{L_2} \Delta E_1.$$

This is the so-called ‘maximin strategy’ for player 1. Similarly if player 2 chooses her leg length first, she will pick the minimax strategy, which solves the following optimization problem:

$$\min_{L_2} \max_{L_1} \Delta E_1,$$
or equivalently, \[
\max_{L_1} \min_{L_2} \Delta E_2.
\]

When each player picks a unique strategy deterministically as above, they are called ‘pure strategies.’

For our bouncing game, the pure maximin and minimax strategies are shown by vertical blue and red lines in Figure 6b and 6c. The maximin and maximin (the vertical lines) happen to coincide at \((L_1, L_2) = (0, 0)\) for falling from a larger height \((H = 2 \text{ m}, \text{Figure 6b})\); that is, it is best to not extend the legs. The maximin and the minimax strategies do not coincide for a smaller drop height \((H = 0.9 \text{ m}, \text{Figure 6c})\): the maximin strategy for player-1 has a small non-zero \(L_1\), with \(L_2 = 0\); the minimax strategy for player-2 is the mirror image of the maximin strategy (reflected about \(L_1 = L_2\)).

To make sense, pure strategies require the order in which the players choose to be decided \textit{a priori}.

In the next section, we examine solution concepts appropriate for when the two players are forced choose a leg-length simultaneously, independent of the other, and not change the decision during the subsequent free-fall.

**Probabilistic strategies and Nash equilibria**

**Mixed strategies.** When the two players have to pick their strategies (leg lengths) simultaneously, it is no longer optimal to pick a fixed deterministic strategy as above. Instead, it is better to use a probabilistic strategy, called a ‘mixed strategy’ [10,11], in which each player picks randomly from the set of possible strategies (leg lengths), with a particular probability distribution.

Analogous to the deterministic minimax problem for pure strategies, one defines a minimax problem over mixed strategies: finding the probability distribution for player-1’s leg lengths that maximizes the \textit{expected value} of \(\Delta E_1\) for player 1 and minimizes the expected value of \(\Delta E_1\) for player 2, and vice versa. The minimax theorem due to von Neumann [11] shows, remarkably, that the solutions to maximin and the minimax problems over mixed strategies are equivalent, and one obtains the same optimal mixed strategy for each player by solving either problem.

Using the discrete payoff matrix, the computation of the optimal mixed strategy can be reduced to the minimization of a linear function subject to linear equality and inequality constraints (see [13]), described in greater detail in Materials and Methods. The unknowns in this linear optimization problem are the probabilities at which the various \(L_1\) and \(L_2\) values are chosen by the respective players.

**Symmetric games. No expected gain.** When all parameters are symmetric, including equality of the initial heights, perhaps not surprisingly, neither player can have a strategy (pure or mixed) that guarantees a non-zero expected energy gain. That is, the expected value \(\Delta E_1 = 0\), and both players will, on average bounce back up to the same height. Indeed, a standard theorem for symmetric zero sum games states that the expected value for minimax mixed strategies is zero [11].

When the initial heights both equal 2 m, the optimal mixed strategy coincides with the pure strategies, namely \((L_1, L_2) = (0, 0)\). For a lower initial height (both equal 0.9 m), when the pure strategies did not coincide (Figure 6c), the optimal mixed strategy for the two players, namely, the probability distribution over the allowed leg lengths, is shown in Figure 6d.

**Asymmetric games. Rich get richer.** We also considered dropping from slightly different initial heights, holding all other parameters symmetrically. In our numerical experiments for a few different initial heights, we found that the mixed minimax solution always had the following property: the player that initially has a higher energy (drops from a higher initial height), had an expected energy gain. In other words, if player-1 has slightly higher energy, the expected value of \(\Delta E_1\) was positive. That is, the rich appear to get richer. This means that any initial energy asymmetry only grows in time, making symmetry unstable even in the game-theoretic sense.
Figure 9. a) Twenty five balls are dropped from approximately but not exactly the same height. b) The angle of the string is shown as a function of time. Note that the string oscillates macroscopically initially, as the masses bounce together coherently on it. However, eventually this macroscopic coherent motion of the masses and the coherent oscillatory string motion gets ‘damped out’ with the energy getting transferred to incoherent motion of the masses. c) A ‘macroscopic’ kinetic energy computed as $m v_{\text{mean}}^2/2$ with mean ball velocity $v_{\text{mean}}$ decreases with time, as the masses’ velocities cancel each other more.

Nash equilibria. Finally, in addition to the minimax paradigm, there is another game theoretic notion called the Nash equilibrium which captures a idea of ‘stability’ — it is the ordered pair of strategies (pure or mixed) such that each player cannot improve her value by unilaterally changing her strategy, as the other keeps her strategy fixed. For zero sum games, the mixed Nash equilibrium is identical to the mixed minimax strategy [11], which we have already characterized in the earlier paragraphs.

Many balls bouncing

As an aside for future work, we generalized our two-ball simulation to the the bouncing of $N$ balls on the trampoline ($N \geq 1$). While we did not examine this system in great detail, we found that even with small numbers of balls, say about 10, the system starts to exhibit properties reminiscent of macroscopic statistical mechanical systems. For instance, Hamiltonian systems with very large number of degrees of freedom, even though energy-conservative, are capable of phenomena analogous to ‘damping’ — conversion of macroscopically observable kinetic energy to internal degrees of freedom. Figure 7 demonstrates this phenomenon in a 25 ball system. Here, initially the macroscopically coherent motion of the masses, moving together as a whole, gets converted into largely incoherent motion of the individual particles. Thus, we see that the oscillation amplitude of the string decays, even through the total energy in the masses is conserved. The mean velocity of the masses (center of mass speed) decreases, while the kinetic energy relative to the system’s center of mass increases. See Materials and Methods for simulation details.
Discussion

In this article, we have examined the mechanics of energy transfer between two masses bouncing on a trampoline and various aspects of their corresponding dynamics. We also examined a game theoretic version of the problem. In the purely dynamical version of the model, we found that for typical and otherwise symmetric parameters, the person that lands second takes off with greater energy. In the game-theoretic version of the problem, with legs that can be extended during flight, we found that an initial asymmetry in energy can lead to even more energy gains for the player with initially greater energy.

The game-theoretic analysis presented here seems unique in that we do not know of a similar analysis of any game involving ‘direct mechanical interaction’ between human players. The classical examples of continuous games are pursuit problems (often differential games), which typically do not involve direct mechanical interaction between the pursuer and the pursued [14]. Of course, there are other examples with dynamical interaction between players (even if not mechanical) in other aspects of human behavior e.g., in economics.

The central dynamical system analyzed here, while energy-conservative, is not Hamiltonian, but only piecewise Hamiltonian. Other non-Hamiltonian energy conservative systems, such as an ideal bicycle [20, 21], which is non-holonomic, and a spring-mass model of human and animal running [14], which is non-smooth (and involves some non-passive but energy-neutral external control), have partial asymptotic stability that Hamiltonian systems cannot have. Perhaps there are variants of our trampoline bouncing model with similar properties ⁴. For instance, we conjecture that it may be possible to achieve (partial) asymptotic stability of symmetric bouncing, by actively changing leg lengths by the two players during flight, but without ever changing the total energy of the system. This would be an example of “cooperative control” [16], albeit in an unconventional energy conservative setting. Indeed, it would be interesting to see if humans can stabilize symmetric bouncing. There is an Olympic sport called synchronized trampoline, in which two gymnasts perform the same routine but on neighboring trampolines. Presumably, attempting this sport on the same trampoline would cause the gymnasts’ performance degrade substantially, because of the sensitive dependence of the bounce on the contact timings. But perhaps this would be a way to detect asynchrony that may be harder to perceive with the naked eye.

The phenomenon of dramatic energy transfer considered here persists even when the trampoline’s stiffness is very large and the contact duration very short (our table-top demonstration is already near this limit). In this limit, the dramatic energy transfer between the two masses through the trampoline is reminiscent of results in the literature on simultaneous ‘rigid-body collisions.’ Simultaneous collisions of nominally rigid-bodies – collisions in which there are multiple points that are making contact at the same time – are known to be often ill-posed. That is, when these simultaneous collisions are ‘regularized’ either by making the contacts happen in sequence or by making the collisions last a non-infinitesimal period with varying time-overlaps between the various contacts, it is found that the details assumed (either the collision sequence or the overlap details) substantially affect the collision consequence [17–21]. The current article provides another example of such sensitivities.

We finally comment on possible implications to the bouncing games that children and adults play on trampolines. The dramatic energy transfer between players bouncing on a trampoline may have an effect of trampoline injuries. Some epidemiological studies [22–24] suggest that a substantial fraction of trampoline injuries are when multiple people were on the trampoline simultaneously. Indeed, official USTA safety manual [25] recommends that more than one person never bounce on the same trampoline. It is possible that the greater injury likelihood with many players on the trampoline could be due to three inter-related effects. First, of course, the large energy transfers between people can produce high

⁴However, in contrast to these two non-Hamiltonian examples, which do not satisfy Liouville’s theorem of phase-space volume preservation [15], the current system does satisfy volume preservation even though only piecewise Hamiltonian. This volume preservation property follows from the fact that (1) within each phase, a volume of initial conditions does not change volume (2) the transitions from phase to another happen with no change in state and therefore, the volume flux across a phase boundary is entirely dependent on the dynamics of the parent phase, which is already volume preserving.
bounces that may be harder to control, resulting in injuries. Second, the larger energy transfers are also
associated with larger forces on the body, often twice as much as when only one person is bouncing.
Third, as is clear from the Figures 3 and 10, the amount of energy transfer can depend very sensitively
on the contact time-lag, lowering the player’s ability to brace for and control the bounce.

Materials and Methods

**Non-smooth dynamics simulations.** The equations of motion (Eq. 1) describing two balls bouncing
were integrated in MATLAB using standard ordinary differential equation solvers (ode45 and ode15s with
high accuracy specifications $\sim 10^{-9}$). Switching from one phase to another phase of the hybrid system is
achieved with the in-built ‘event detection’ capabilities of the ODE solvers, so that the solution is formed
as a patch-work of solutions to smooth differential equations. See Supplementary Information for the
complete MATLAB code.

For $N$ ball simulations, $N > 2$, we now present the equations of motion in an algorithmic form. Let
$(x_i, y_i)$ denote the horizontal and vertical positions of the $i$th ball, $i = 1 \ldots N$. If ball-$i$ is not in contact
with the string, its motion is governed by $\ddot{y}_i = -g$. At any moment, the set of balls that are in contact
with the string are determined as those with $y < 0$. Say there are $m$ balls in contact and the set $p$
contains the indices of the balls in contact. For instance, if balls 3, 5, and 8 are in contact, $p = \{3, 5, 7\}$
and $p(2) = 5$.

The equation for $j$th ball in contact $(j = 1 \ldots m)$ is:

$$\ddot{y}_j = -g + \frac{T}{m} \left( \frac{y_{p(j+1)} - y_{p(j)}}{x_{p(j+1)} - x_{p(j)}} + \frac{y_{p(j+1)} - y_{p(j)}}{x_{p(j+1)} - x_{p(j)}} \right)$$

in which, in addition to the above definitions, we have $p(0) = 0$, $p(m + 1) = N + 1$. Also, $(x_0, y_0)$
and $(x_{N+1}, y_{N+1})$ are the fixed positions of the left and right ends of the string. See Supplementary
Information for the complete MATLAB code. For this ODE solution, we simply integrate over the non-
smoothness in the right-hand side of the differential equation, using the stiff solver ode15s. While this
is not ideal for solution accuracy (as the solvers assume higher-order differentiability), the adaptive step-
sizing keeps the error small enough that energy is conserved over reasonable time-scales. This accuracy
seems sufficient for obtaining overall qualitative or statistical properties of the dynamics (as verified by
comparing with the more accurate two-ball simulation).

**Table-top experiment.** For the physical table-top experiment, we used lacrosse and tennis balls
dropped onto Gold’s Gym mini trampoline, 36 inches in diameter. We did not tune any of the stiffnesses,
and our calculations suggest that the effects will be seen on trampolines of vastly different mechanical
properties.

**Human experiments.** Even though our article is about a model for people bouncing on trampolines,
we performed no human subject experiments, nor any analysis of any already collected human data.

**Computing game-theoretic solutions.** The deterministic minimax or maximin solutions of the zero
sum game, given a pay-off matrix, are simply computed by looping through the rows and columns, and
determining, for instance, the minimum of all the row maximums. The mixed equilibria, as noted earlier, are solved using linear programming, briefly described as follows. Consider the perspective of player-1, who wishes to maximize $\Delta E_1$. As noted earlier, we discretize
the continuous space of strategies, so that we can examine finitely many strategies $L_1(i)$ and $L_2(j)$ chosen
by the two players ($i = 1 \ldots q, j = 1 \ldots q$, say). Say the corresponding payoff matrix is $A$, with elements
\[ A_{ij} = \Delta E_1(L_1(i), L_2(j)) \] and player-1 chooses strategy \( L_1(i) \) with \( p_i \). These \( p_i \) are found by solving the following linear optimization problem [13, 26]:

\[
\begin{align*}
\text{maximize} & \quad u \\
\text{such that} & \quad p_1 + \cdots + p_q = 1, \quad 0 \leq p_i \leq 1. \\
& \quad u \leq \sum_{i=1}^{q} p_i A_{i1} \\
& \quad \vdots \\
& \quad u \leq \sum_{i=1}^{q} p_i A_{iq}
\end{align*}
\]

This is a linear programming problem solved using standard software (`linprog` in MATLAB). Again, the MATLAB code used to solve this problem is part of the supplementary material.

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### Appendix

#### Energy transfer: some limiting cases

In the section, we have characterized to some extent (Figures 3a-c), the dependence of the post-bounce state on the details of the pre-bounce state.

**Large energy “collisional” limit.** When the initial (pre-contact) kinetic energy of the two masses is large, many of the parameter dependences vanish, resulting in some simplification. In this limit, which is the same as the tension in the string being very high, we note that there is: (1) no dependence on gravity (equivalent to setting \( g \to 0^+ \)), as the gravity would be much smaller than the elastic forces during contact; (2) no dependence on the masses, only their ratios. (3) no dependence on the stiffness, only the length ratios \( a/L \) and \( b/L \); (4) no dependence on the initial energies (or initial heights), only their ratios.

Say the masses are equal, symmetrically placed on the trampoline, and start with the same large energy. Then, the relative energy transfers depend only on the length ratio \( b/L \) (Figures 10a-c). Our claim of independence with respect to various parameters in this large energy limit is buttressed by the fact that each of these figures is a superposition of up to 9 different curves, and for many parameter values, varying by more than an order of magnitude or two.

Complete energy transfer seen in Figures 3a-c is not found in this limit (Figures 10a-c); we do not have a simple explanation. When the two masses are far apart, placed near the extremes of the string, their interaction through the string reduces and the maximum energy transfer goes to zero as \( b \to L \) (Figure 10c).
Figure 10. We show the relative energy transfer for the large energy limit, when gravity is not an issue, nor are many other parameters. a) $b = L/3$. b) The two masses very close together $b = 0.002L$. The two masses go through many oscillations before both of them go to flight phase. c) The two masses far apart. There is little interaction. When $b \approx L$, the relative energy transfer goes to zero. d) When $m_2 = 0.001m_1$, $b = L/3$ and initial drop heights ($H_1 = H_2$, not energies) are equal, the relative change in energy of $m_1$ is essentially zero, but the relative change in the energy of $m_2$ can be up to a factor of 3.

Figure 11. When the two masses are very close together, the mass in contact leaves contact as soon as the second mass makes contact. Thus the entire elastic energy of mass-1 is lost by mass-1, and transferred to mass-2. The two figures show energy transfer at the point when one of the two masses takes off.

Two masses very close together. When the two masses are very close to each other, if mass-2 makes contact when mass-1 is in contact, mass-1 loses contact almost immediately. This is because, mass-2 pulls the string down — and because of its closeness to mass-1, it has to travel a very small distance downwards before the string around mass-1 becomes straight, allowing mass-1 to take off. Often, before both masses are simultaneously in flight (that is, phase P0 is reached), the two masses go through many P1 and P2 phases, interchanging contact, especially when the contact time-lag was very small. The energy transfer until phase P0 is reached, at the high energy limit, is shown in Figure 10b; the oscillations in the energy transfer obtained reflects the multiple phases gone through until P0.

Instead of plotting the energy transfer when phase P0 is reached, it is more illuminating in this case to plot the energy transfer at the moment when one of the balls has taken off (Figure 11a). If mass-1 leaves contact immediately when mass-2 lands, then the energy transferred to mass-2 is whatever was left in the string when mass-2 contacted. This expected equality of the energy transferred to mass-2 and the elastic energy in the string when mass-2 contacted is shown in Figure 11b.

Very different masses. A classic physics demonstration involves placing a tennis or ping pong ball on top of a basketball, and dropping them simultaneously from rest. The dramatic result is that the smaller ball bounces to many times its initial height, sometimes hitting the room’s ceiling. This behavior
is understood by noting that the basketball’s velocity is reversed first, and that the tennis ball is colliding with a basketball that is already moving upward. For elastic collisions, and in the limit of zero mass ratio, the smaller ball rises to nine times its initial height. See [27,28] for a careful derivation.

Analogously, say $m_2 \ll m_1$, mass-2 makes contact when mass-1 is already in contact, and that the two masses have been dropped from rest from roughly the same initial heights, so the initial energies are very different. For $b = c = L/3$, assuming that mass-2’s bounce is short and does not affect mass-1’s velocity, we can show that: (1) mass-2 gains most energy if it contacts mass-1 has maximum upward speed, about to take-off; mass-2’s energy increase is three times its initial energy. (2) mass-2 loses essentially all its energy if it lands just after mass-1 has landed, when it has the greatest downward speed. These two limits are reflected in Figure 12a, when the contact time-lag is, respectively, close to one or zero.

When the initial energies are equal, $m_2 \ll m_1$, mass-2’s bounce now has an effect on mass-1’s dynamics, so the above simple intuition falls apart. Energy transfer in this case and in the case when the mass landing second $m_2 \gg m_1$ are shown in Figure 12b and 12c respectively.

![Figure 12](image_url)

**Figure 12.** Very different masses. a-b) Lighter mass $m_2$ lands second. a) Dropped from the same height. b) Dropping with similar initial energies (different heights). c) Heavier mass $m_2$ lands second.

### References


**Figure Legends**

**Figure 1.** A two-person game on a trampoline: Seat drop war. Only one player shown. Each player alternatively bounces with her feet and her ‘seat’. The sequence of body configurations for one player is shown schematically. The other player goes through a similar sequence of configurations, but possibly with a phase difference.

**Figure 2.** Simple model of two people or two balls bouncing on a trampoline, as two point-masses on a massless trampoline. a) The system can be in one of four phases: neither mass in contact with the trampoline (P0), only mass-1 in contact (P1), only mass-2 in contact (P2), and both masses in contact (P12). b) The geometry of the system is shown, along with the forces on the masses when both are in contact with the trampoline.

**Figure 3.** Energy transfer and contact time-lag. Energy increase in mass-1 (red) and mass-2 (blue) as a function of the impact time-lag. The masses are equal and are placed symmetrically on the trampoline, so that $a = c = (L - b)/2$. The separation between the masses $b = L/3$. a) The initial energies $E_1(0) = E_2(0)$ are equivalent to dropping from rest from a height of 1 m. b) Mass-1’s initial energy $E_1(0)$ is the same as for panel-a, but $E_2(0) = 2E_2(0)$. c) Mass-1’s initial energy $E_1(0)$ is the same as for panel-a, except mass-2’s energy is $E_2(0) = E_1(0)/2$. 
Figure 4. The basic mechanism of energy transfer from mass-1 to mass-2. When mass-1 is moving up, the presence of mass-2 lowers the work done by the string on mass-1. Thus mass-1 takes off with lesser upward velocity than if mass-2 had not interfered.

Figure 5. The eigenvalues of a return map on the periodic orbit corresponding to a symmetric bounce for symmetric masses. The product of the eigenvalues was equal to 1 (with an error of about $10^{-4}$). In the intermediate regime shown, all three eigenvalues, two of which are complex conjugates and reciprocals of each other, have unit absolute values.

Figure 6. Bouncing as a zero sum game. a) Two players drop from the same height and they can pick a rigid leg length to modify their contact times. b) The normalized energy increase in mass-1 (the payoff function) as a function of the leg length choices of the two players, when the two players drop from an initial height of $H = 2$ m. The vertical line (blue/red) shows the minimax and maximin strategy that coincide. c) The payoff function when initial height is $H = 0.9$ m. For this height, the pure minimax (denoted player-2) and maximin (denoted player-1) strategy do not coincide. The second panel shows the optimal probabilities corresponding to mixed minimax strategies.

Figure 7. a) Twenty five balls are dropped from approximately but not exactly the same height. b) The angle of the string is shown as a function of time. Note that the string oscillates macroscopically initially, as the masses bounce together coherently on it. However, eventually this macroscopic coherent motion of the masses and the coherent oscillatory string motion gets ‘damped out’ with the energy getting transferred to incoherent motion of the masses. c) A ‘macroscopic’ kinetic energy computed as $m v_{\text{mean}}^2/2$ with mean ball velocity $v_{\text{mean}}$ decreases with time, as the masses’ velocities cancel each other more.