Harmonic balance

Given some nonlinear ODE

\[ \dot{x} = F(t, x) \]  

\[ \Rightarrow \text{assume that the solution is periodic} \]

\[ \Rightarrow \text{assume that the solution is given by a Fourier series} \]

\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega t) + b_n \sin(n\omega t) \right] \]

\[ \Rightarrow (\text{Generally}) \text{ retain only } N \text{ terms of the Fourier series} \]

\[ \Rightarrow \text{so that we use } n = 1 \ldots N. \]

\[ \Rightarrow \text{Substitute } 3 \text{ in } 1. \]

\[ 1 \quad \dot{x} - F(t, x) = 0 \]

\[ \Rightarrow \dot{x}(t) - F(t, \dot{x}(t)) = 0. \]

Of course this equation will not be exactly satisfied because we have truncated the series to only \( N \) terms.
This method does not necessarily work well.

Better would be to use much more than 2N+1 points and try to minimize the sum of squares of residuals at those points using either fsolve or fminunc.
Method 2. "Galerkin projection"

The residual $g(t)$ projected on to the basis functions $1, \sin(nwt), \cos(nwt)$ for $n = 1 \ldots N$ = 0.

So that the only residuals remaining are of higher frequency than $N wt$.

i.e., we set up the following $2N+1$ equations and solve for $2N+1$ unknowns $a_0, a_1 \ldots a_N, b_1 \ldots b_N$.

\[ \frac{2\pi}{\omega} \int g(t) \, dt = 0 \quad \leftarrow 1 \text{ equation} \]

\[ \frac{2\pi}{\omega} \int g(t) \cos(nwt) \, dt = 0 \quad \leftarrow N \text{ equations} \]

\[ \frac{2\pi}{\omega} \int g(t) \sin(nwt) \, dt = 0 \quad \leftarrow N \text{ equations} \]

This projection of $g(t)$, the residual, on to the basis $\sin(nwt)$.
We get $2N+1$ equations by setting all terms until $\sin(Nwt)$ and $\cos(Nwt)$ to zero.

The above discussion assumes that we know the period $\frac{2\pi}{\omega}$, for instance, knowing a forcing frequency $\omega$.

If we do not know the period $T = \frac{2\pi}{\omega}$, we have $2N+2$ unknowns with $2N+1$ equations.

In this case, we may have one free parameter, e.g., $\omega$ as a function of one of the coefficients or amplitude (or/and) determined by initial conditions.

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**Note:** The Galerkin projection is the SAME as expanding all terms (powers of cosines and sines)

Using $\cos^2 w t = \frac{\cos 2w t + 1}{2}$

$\cos^3 w t = \frac{3 \cos w t + \frac{1}{4} \cos 3w t}{4}$

$\cos^k w t = \ldots$ [Fourier series of this function]

And then using only terms until $\cos N w t$. 
Duffing equation (no forcing, no damping).

\[ \ddot{x} + x + \epsilon x^3 = 0. \quad (1) \]

ideally, we should include \( A_0 + B \sin wt \).

\[ x = A_0 + A_1 \cos wt + B_1 \sin wt. \quad (2) \]

\( A_0, A_1, B_1, \) \( w \) - all unknowns.

We know that \( x(t) \) is oscillatory.

\( \Rightarrow \) there is some point where \( x = 0 \) take that to be \( t = 0 \).

\( \Rightarrow \) without loss of generality we can assume \( x = A_0 + A_1 \cos wt. \quad (3) \)

Use \((3)\) in \((1)\)

\[ \ddot{x} = -A_1 w^2 \cos wt \]

\[ x = A_0 + A_1 \cos wt \]

\[ \epsilon x^3 = \left( A_0 + A_1 \cos wt \right)^3 \]

\[ = \epsilon \left[ A_0^3 + A_0^2 A_1 \cos^2 wt + 3A_0 A_1^2 \cos^2 wt + 3A_0^2 A_1 \cos^2 wt + 3A_0 A_1^2 \cos^2 wt \right] \]

Identities:

\[
\begin{align*}
\cos 3\theta &= 4\cos^3 \theta - 3\cos \theta \\
\cos^3 \theta &= \frac{\cos 3\theta + 3\cos \theta \cdot 0}{4}
\end{align*}
\]
\[ \varepsilon X^3 = \varepsilon A_0^3 + \varepsilon A_1^3 \left[ \frac{\cos 3 \omega t + 3 \cos \omega t}{4} \right] + \varepsilon 3 A_0^2 A_1 \cos \omega t + \varepsilon 3 A_0 A_1^2 \left[ \frac{\cos 2 \omega t + 1}{2} \right]. \]

Keep only terms involving constant + \cos \omega t.

Ignore \cos 2 \omega t, \cos 3 \omega t, etc.

\[ \varepsilon X^3 = \varepsilon A_0^3 + \cos \omega t \left[ \frac{3}{4} \varepsilon A_1^3 \right] + \varepsilon 3 A_0 A_1^2. \]

Put everything together

\[ X + x + \varepsilon X^3 = 0. \]

\[ -A_1 \omega^2 \cos \omega t + A_0 + A_1 \cos \omega t + \varepsilon A_0^3 + \varepsilon 3 A_0 A_1^2 + \cos \omega t \left[ \frac{3}{4} \varepsilon A_1^3 \right] = 0. \]

\[ \Rightarrow A_0 + \varepsilon A_0^3 + \varepsilon 3 A_0 A_1^2 = 0 \quad \Rightarrow \quad A_0 = 0 \quad (\text{or } A_1 \text{ is imaginary}). \]

\[ \cos \omega t \left[ -A_1 \omega^2 + A_1 + \frac{3}{4} \varepsilon A_1^3 \right] = 0. \]

\[ A_0 \neq 0 \quad \Rightarrow \quad \omega^2 = 1 + \frac{3}{8} \varepsilon A_1^2. \]

[\( \omega \approx 1 + \frac{3}{8} \varepsilon A_1^2 \)]
Solution

\[ X(t) = A_1 \cos \omega t \]

where \( \omega = \sqrt{1 + \frac{3\varepsilon A_1^2}{4}} \)

\[ \approx 1 + \frac{3\varepsilon A_1^2}{8} \]

amplitude dependence of frequency.

\[ \]

\underline{Shorter version}

Assume \( X(t) = A_1 \cos \omega t \)

\[ \ddot{X} = -A_1 \omega^2 \cos \omega t \]

\[ X^3 = A_1^3 \cos^3 \omega t \]

\[ = A_1^3 \left[ \cos 3\omega t + 3 \cos \omega t \right] \]

\[ \ddot{X} + X + \varepsilon X^3 = 0, \]

\[ -A_1 \omega^2 \cos \omega t + A_1 \cos \omega t \varepsilon A_1^3 \left[ \frac{3 \cos \omega t}{4} + \frac{\cos 3\omega t}{4} \right] = 0 \]

\[ \]  

\[ \cos \omega t \left[ -A_1 \omega^2 + A_1 + \frac{3}{4} A_1^3 \varepsilon \right] = 0 \]

\[ A_1 \neq 0 \Rightarrow \left[ \omega^2 = 1 + \frac{3}{4} \varepsilon A_1^2 \right] \]

\[ \]
Example 2

\[ x' + cx + x + ax^3 = F \sin \omega t \]  \hspace{1cm} (1)

Assume

\[ x = A \sin \omega t + B \cos \omega t \]  \hspace{1cm} (2)

\[ \Rightarrow \text{Use (2) in (1), Expand, Use trig identities,} \]

\[ \text{for } \sin^3 \omega t, \cos^3 \omega t, \sin^2 \omega t, \cos^2 \omega t \] \text{ powers in terms of } \sin \omega t, \cos \omega t, \sin 2\omega t, \cos 2\omega t, \sin 3\omega t, \cos 3\omega t.

\[ \Rightarrow \text{set } \sin \omega t \text{ and } \cos \omega t \text{ terms to zero.} \]

we get

\[ -A\omega^2 + CBw + A + \frac{3}{4} \alpha A^3 + \frac{3}{4} \alpha AB^2 - F = 0 \]

\[ -B\omega^2 + CAw + B + \frac{3}{4} \alpha A^2B + \frac{3}{4} \alpha B^3 = 0 \]

\[ \text{forcing} \]

for given \( \omega \) and \( F \), we can solve for \( A \) and \( B \)

and therefore the response amplitude

\[ \sqrt{A^2 + B^2} \] gives the frequency response
We can solve these equations analytically or numerically using `fsolve` to obtain the frequency response curves for the Duffing oscillator.
Example 3

**Limit cycle oscillators**

Ray. Vnder Pol. oscillator

\[ \dot{x} + \mu (x^2 - 1) \dot{x} + x = 0 \]  

\[ (\text{or}) \quad \text{Rayleigh oscillator,} \quad \ddot{y} + \gamma = e (\dot{y} - \frac{y^3}{3}) , \]  

\[ \rightarrow \text{assume } x = A \cos \omega t . \]

\[ \text{Use (3) or (4) in (1) or (2)} \]

\[ \text{already assuming } x(t) \text{.} \]

\[ \text{need not assume this} \]

\[ \text{to solve for } A \text{ and } \omega \]

the two equations you will need for these two unknowns will come from equating to zero, the coefficients of \( \cos(\omega t) \) and \( \sin(\omega t) \)

**Warning:**

While these simple examples all yield useful results with the simple assumption that \( x(t) = A \cos(\omega t) \) one should be careful.

It is good to use at least one higher Fourier order to make sure that the higher order terms do not change the solution dramatically, or check numerically that the solutions are at least qualitatively correct.