STABILITY OF FIXED POINTS (EQUILIBRIA) OF NON-LINEAR SYSTEMS

- For first systems, we found that finding all the fixed points and determining their stability gives a lot of information about the overall dynamics of the equation for various initial conditions. Similarly, for higher dimensional systems \( \dot{y} = f(y) \), it is usually useful to identify all the fixed points/equilibrıa of the system and determine their stability.

We usually follow these steps:

1. Find all fixed points \( y^* \) such that \( f(y^*) = 0 \) (requires the solution of nonlinear equations).

2. Linearize the system about the fixed points.

3. Find eigenvalues of \( J \), which usually (not always) provide information about the stability of the fixed points.

These steps are explained in more detail in the next few pages and the next few classes.
FIXED POINTS AND LINEARIZATION OF \( \frac{dy}{dt} = f(y) \) — (1)

**FIXED POINT**: \( y^* \) is a fixed point of (1) if \( f(y^*) = 0 \).

\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{bmatrix},
\]

\[
f(y) = f(y^*) + J(y - y^*) + O(\|y - y^*\|^2).
\]

**LINEARIZATION**

where \( J \) = Jacobian of \( f \) with respect to \( y \).

\[
J_{ij} = \frac{\partial f_i}{\partial y_j} \quad \text{evaluated at } y^*.
\]

Matrix \( J_{ij} = \frac{\partial f_i}{\partial y_j} \). Say \( z = y - y^* \), then \( \frac{dz}{dt} = Jz \) is the linearization

**Example 1**: 1 degree of freedom, nonlinear damped spring mass system.

\[
\ddot{x} + h(x, \dot{x}) = 0
\]

\[
\begin{align*}
y_1 &= x \\
y_2 &= \dot{x}
\end{align*}
\]

\[
\begin{align*}
y_1 &= y_2 \\
y_2 &= -h(y_1, y_2)
\end{align*}
\]

**FIXED POINT / EQUILIBRIUM**: \( y_1 = 0 \) and \( h(y_1, y_2) = 0 \)
Example 2: Simple pendulum

\[ m \ell^2 \ddot{\theta} + mg \ell \sin \theta = 0 \]

\[ \ddot{\theta} + g \frac{\sin \theta}{\ell} \theta = 0 \]

Fixed point: \( \ddot{\theta} = 0 \)
\[ \theta = \frac{m \ell^2}{\ell} \sin \theta = 0 \]
\[ \theta = \left\{ \begin{array}{l}
\theta = 0 \\
\theta = \pm \pi \end{array} \right. \]

Alternatively,
\[ x_1 = \theta \quad x_2 = \dot{\theta} \]
\[ \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_1 = x_2 \\
\dot{x}_2 = -g \sin x_1 - cx_2
\end{bmatrix} \]

Fixed points: \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \)
\[ x_1 = 0 \quad x_2 = 0 \]

So fixed points \((0, 0)\) and \((\pm \pi, 0)\).

Linearization about fixed points
\( f_1(x_1, x_2) = x_1 \)
\[ f_2(x_1, x_2) = -cx_2 - \frac{g}{\ell} \sin x_1 \]

\[ J = \begin{bmatrix}
0 & 1 \\
-\frac{g}{\ell} \cos x_1 & -c
\end{bmatrix} \]

About \((0, 0)\)
\[ J \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) =
\begin{bmatrix}
0 & 1 \\
-\frac{g}{\ell} & -c
\end{bmatrix} \]
\[ z = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \]

About \((\pi, 0)\)
\[ J \left( \begin{bmatrix} \pi \\ 0 \end{bmatrix} \right) =
\begin{bmatrix}
0 & 1 \\
-\frac{g}{\ell} & -c
\end{bmatrix} \]
\[ z = \begin{bmatrix} 0 \\ +1 \end{bmatrix} \]
CAUTION: STABILITY OF FIXED POINTS OF ODEs in N Dimension

\[ \dot{X} = f(X) , \]
\[ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \]

- Find a fixed point \( X^* \).
- Find the Jacobian of \( f(X) \) w.r.t \( X \) or \( X^* \).
  
  \[ J = \text{some } N \times N \text{ matrix} \]
- Find the eigenvalues of \( J \).
  - if all eigenvalues have negative real parts, then \( X^* \)
    is asymptotically stable.

Note if \( N \geq 2 \) i.e., 3, 4, ...
the trace \( T \) and the determinant \( \Delta \) are \( \text{not} \)
sufficient information to determine stability.

You have to find the \( N \) eigenvalues by setting up
the characteristic equation by

\[ \det (J - \lambda I) = 0 \]

\( N \) th degree polynomial in \( \lambda \)
\( N \) eigenvalues, real or complex.

\[ X = AX \]

\[ A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -10^{-5} \end{bmatrix} \]

\( \text{UNSTABLE} \) because \( \lambda_3 = 10^{-5} > 0 \)

Note: here \( \lambda < 0 \)
and \( \Delta > 0 \)
but still
unstable.
Linear Stability Analysis

Find the Jacobian $J$ of the RHS at equilibrium point $x^*$

If all eigenvalues of $J$ have negative real parts, the equilibrium is asymptotically stable.

If even one eigenvalue of $J$ has a positive real part, the equilibrium is unstable.

If some eigenvalues have negative real parts and other eigenvalues have zero real parts, then the Jacobian does not have enough information about the stability of the equilibrium if the system is nonlinear. In this situation, the equilibrium can be asymptotically stable, unstable, or "neutrally stable" (sort of hanging around without converging to the equilibrium or going too far away).

Later on, we will discuss precise definitions of various notions of stability: Lyapunov stability, asymptotic stability and exponential stability.