General 2nd order ODE: (LINEAR)

\[ \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = 0 \]

**Fixed Point:** \( x = 0 \).

Hypothesis for solution: \( x = ve^{\lambda t} \) \( \dot{x} = v\lambda e^{\lambda t} \) \( \ddot{x} = v\lambda^2 e^{\lambda t} \).

\[ \Rightarrow \frac{\lambda^2}{m} ve^{\lambda t} + \frac{c\lambda}{m} ve^{\lambda t} + \frac{k}{m} ve^{\lambda t} = 0 \]

\[ \frac{\lambda^2 + c\lambda + k}{m} = 0 \]

Two solutions:

\[ \lambda_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m} \]

General solution for \( x(t) \):

\[ x(t) = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t} \]

When positive damping and stiffness
when \( c > 0 \) and \( k > 0 \) \( \Rightarrow \) \( x = 0 \) is stable. (\( x \) decays to \( x = 0 \))

When \( c^2 - 4km > 0 \), \( \sqrt{c^2 - 4km} \) is real and the solution consists of exponentials and no oscillations. (OVERDAMPED)

When \( c^2 - 4km < 0 \), \( \sqrt{c^2 - 4km} \) is complex and the solution consists of sines, cosines and an exponential. => Oscillatory solutions (UNDER DAMPED)

When \( c < 0 \) or \( k < 0 \) atleast one of \( \lambda_1 \) and \( \lambda_2 \) has a positive real part, (negative damping or negative stiffness.

\( x = 0 \) UNSTABLE.

Again, when \( c^2 - 4mk > 0 \), no oscillations \& growth

When \( c^2 - 4mk < 0 \), oscillatory growth
LINEAR SYSTEMS IN 2D

The stability or otherwise of a fixed point / equilibrium of a nonlinear system is "usually" (in a sense that will be made precise) determined by analyzing the linearization.

So we first study linear systems in detail.

General 2D linear system:

\[
\begin{align*}
\frac{dy_1}{dt} &= a_{11} y_1 + a_{12} y_2 \\
\frac{dy_2}{dt} &= a_{21} y_1 + a_{22} y_2
\end{align*}
\]

\[\dot{Y} = AY \quad \text{(1)}\]

General solution for \(Y(t)\)

Hypothesis \(Y = V e^{\lambda t}\) \quad \text{(2)}

where \(V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\)

(2) in (1) gives

\[\lambda V e^{\lambda t} = AVE^{\lambda t}\]

\[AV = \lambda V \quad \text{"Eigenvalue problem"}\]

\[(A - \lambda I) V = 0\]

\(V\) - Eigenvector
\(\lambda\) - Eigenvalue

Finding eigenvalues

For 2D:

\[(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0\]

\[\Rightarrow \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11} a_{22} - a_{12} a_{21}) = 0\]

TRUE for N-DIMENSIONAL SYSTEMS

\[y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\]
There are 2 solutions \( \lambda_1 \) and \( \lambda_2 \).

\[
\lambda^2 - \lambda \left( \text{trace}(A) \right) + \text{det}(A) = 0
\]

\[
\lambda_{1,2} = \frac{-\text{trace}(A) \pm \sqrt{\left[ \text{trace}(A) \right]^2 - 4 \text{det}(A)}}{2}
\]

Note: We showed that \( \dot{y} = Ay \) is equivalent to

\[
\begin{align*}
\ddot{z} + (a_{11} + a_{22}) \dot{z} + (a_{11} a_{22} - a_{12} a_{21}) z &= 0 \\
\ddot{z} - (\text{trace}(A)) \dot{z} + (\text{det}(A)) z &= 0
\end{align*}
\]

- trace(A) = damping/mass (can be negative)
- determinant(A) = stiffness/mass (can be negative)

So general solution is:

\[
y(t) = \text{a linear combination of } V_1 e^{\lambda_1 t} \text{ and } V_2 e^{\lambda_2 t}
\]

\[
= k_1 V_1 e^{\lambda_1 t} + k_2 V_2 e^{\lambda_2 t}
\]

\( k_1 \) and \( k_2 \) are constants that can be determined given initial conditions \( y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \).
Let \( \tau = \text{trace}(A) \)
\( \Delta = \text{determinant}(A) \).

\[
\lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right)
\]

**Case 1:** \( \tau^2 - 4\Delta > 0 \)

\( \Rightarrow \lambda_{1,2} \text{ REAL} \).

Solution looks like \( \gamma(t) = k_1 V_1 e^{\lambda_1 t} + k_2 V_2 e^{\lambda_2 t} \).

(\text{STABLE}). \( \gamma(t) \) decays to \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) in general only when both \( \lambda_1 \) and \( \lambda_2 \) are negative.

\[
\Rightarrow \quad \tau + \sqrt{\tau^2 - 4\Delta} < 0
\]
and \( \tau - \sqrt{\tau^2 - 4\Delta} < 0 \)

\[
\text{or } \tau - \sqrt{\tau^2 - 4\Delta} < 0 \quad \Rightarrow \quad \boxed{\tau < 0}
\]
and \( \tau + \sqrt{\tau^2 - 4\Delta} < 0 \) & \( \tau < 0 \quad \Rightarrow \quad \boxed{\Delta > 0} \)

(\text{UNSTABLE}) if either \( \tau > 0 \) or \( \Delta > 0 \)

In all these cases, because \( \gamma(t) \) consists only of exponentials and no sines and cosines, we have no oscillations.
\[ \text{Case 2: } \gamma^2 - 4 \Delta < 0 \]

\[ \Rightarrow \lambda \pm i \sqrt{4 \Delta - \gamma^2} \quad \text{COMPLEX CONJUGATES.} \]

Solution looks like

\[ y(t) = e^{\gamma t} \left[ k_1 V_1 \cos \sqrt{4 \Delta - \gamma^2} t + k_2 V_2 \sin \sqrt{4 \Delta - \gamma^2} t \right] \]

\[ \text{exponential} \quad \text{(monotonic)} \]

\[ \text{oscillatory} \]

\[ \text{(STABLE)} \quad y(t) \text{ decays if } \gamma < 0 \text{ and } e^{\frac{\gamma t}{2}} \to 0 \text{ as } t \to \infty \]

This decay will be oscillatory.

\[ \text{(UNSTABLE)} \quad y(t) \text{ grows unbounded if } \gamma > 0 \text{ and } e^{\frac{\gamma t}{2}} \to \infty \text{ as } t \to \infty \]

This growth will be oscillatory.
\[ \frac{dY}{dt} = AY \]
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \]

\[ \gamma = \text{trace} (A) = a_{11} + a_{22} \]
\[ \Delta = \text{determinant} (A) = a_{11}a_{22} - a_{12}a_{21} \]

**STABLE**

if and only if \( \gamma < 0 \) and \( \Delta > 0 \).

**UNSTABLE**

if \( \gamma > 0 \) or \( \Delta < 0 \)

**OSCILLATIONS** if \( \gamma^2 - 4\Delta < 0 \) (like underdamped)

**NO OSCILLATIONS** if \( \gamma^2 - 4\Delta > 0 \) (like overdamped)

### SUMMARY

not counting boundary cases.
\[\dot{y} = Ay \quad \Rightarrow \quad \dot{y} = f(y).\]

\[y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\]

In the state variables. It is the set of all variables whose values need to be specified in general so as to be able to determine the future of system for all time.

State Space: set of all values for the state variables.

E.g. if \(y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\) is the state variables, the state space is \(\mathbb{R}^2\), the plane.

In 2D, the state space is also called the phase plane.

In 3D, the phase space is also called the phase space.

In 2D, the qualitative dynamical behavior of the system can be represented graphically by plotting \(y_1(t)\) vs. \(y_2(t)\) [in the phase plane] for a number of initial conditions. Such a figure is called a phase portrait. The individual trajectories \(y_1(t)\) vs. \(y_2(t)\) are called phase trajectories.

In 3D, one can imagine similar trajectories through the 3-dimensional state space.
Example 1

\[ \dot{Y} = \mathbf{A}Y, \quad \mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \]

\[ \lambda_1, \text{ and } \lambda_2 \text{ real.} \]

Eigenvalues \( \lambda_1 \) and \( \lambda_2 \).

Eigenvectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) respectively.

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

General solution \( Y(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 \).

\[ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 \\ \lambda_2 \mathbf{v}_2 \end{bmatrix} \]

\[ y_1(t) = y_1(0) e^{\lambda_1 t}, \quad y_2(t) = y_2(0) e^{\lambda_2 t} \]

A phase portrait is obtained by plotting \( y_1(t) \) vs \( y_2(t) \) for various initial conditions.

Case 1 \( \lambda_1 = \lambda_2 < 0 \)

\[ y_1(t) = y_1(0) e^{\lambda_1 t}, \quad \frac{y_1(t)}{y_2(t)} = \frac{y_1(0)}{y_2(0)} \]

\[ y_2(t) = y_2(0) e^{\lambda_2 t}, \quad \frac{y_2(t)}{y_1(t)} = \frac{y_2(0)}{y_1(0)} \]

That is, \( y_1 \) vs \( y_2 \) is a straight line with slope \( \frac{y_2(0)}{y_1(0)} \).

The slope depends on initial condition and the line passes through the origin.

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ called a "STABLE NODE" in particular a "STABLE STAR NODE"} \]
The direction of the arrows indicate what happens to \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) as time moves forward. Here, wherever you start (whatever the initial condition), \( \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) as \( t \to \infty \).

**Case 2**

\( \lambda_1 < \lambda_2 < 0 \)

\[
\begin{align*}
y_1(t) &= y_1(0) e^{\lambda_1 t} \\
y_2(t) &= y_2(0) e^{\lambda_2 t}.
\end{align*}
\]

\( \lambda_1 > \lambda_2 \Rightarrow y_1 \) decays faster than \( y_2 \), which is reflected in the phase portrait below:

The fixed point \([0,0]\) is called a **STABLE NODE**. (not a STAR though)

**Case 3**

\( \lambda_2 < \lambda_1 < 0 \)

Again called a **STABLE NODE**.

Here \( \lambda_2 < \lambda_1 < 0 \Rightarrow y_2 \) decays faster than \( y_1 \).
**Case 4** \( \lambda_1 = \lambda_2 > 0 \)

**UNSTABLE STAR NODE**

\( y_1 \) and \( y_2 \) grow at the same rate near the fixed point.

**Case 5** \( \lambda_1 > \lambda_2 > 0 \)

**UNSTABLE NODE**

\( y_1 \) grows faster than \( y_2 \).

**Case 6** \( \lambda_2 > \lambda_1 > 0 \)

**UNSTABLE NODE**

\( y_2 \) grows faster than \( y_1 \).

**Case 7** \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \)

\( y_1(t) = y_1(0) e^{\lambda_1 t} \to \infty \)

\( y_2(t) = y_2(0) e^{\lambda_2 t} \to 0 \)

The FP is called a **SADDLE POINT**.

The phase portrait is consistent with the following observations.

If you start on the \( y_1 \)-axis, i.e., \( \begin{bmatrix} y_1(0) \\ 0 \end{bmatrix} \), \( y_1(t) \to \infty \) and \( y_2 \to 0 \).

If \( y_1(0) > 0 \), \( y_1(t) \to -\infty \) and \( y_2 \to 0 \).

If \( y_1(0) < 0 \), \( y_2(t) \to 0 \).

Similarly, if you start on the \( y_2 \)-axis, i.e., \( \begin{bmatrix} 0 \\ y_2(0) \end{bmatrix} \), \( y_1(t) \to 0 \).
Here, the $y_1$-axis is called the **UNSTABLE MANIFOLD**.

More generally, the eigenvector corresponding to the real eigenvalue that is positive gives the **UNSTABLE MANIFOLD**.

Here it is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here, the $y_2$-axis is called the **STABLE MANIFOLD**.

For a more general saddle point, the **STABLE MANIFOLD** is given by the eigenvector corresponding to the negative real eigenvalue.

Here, given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Remarks:**

1. In nonlinear systems, the stable manifold of a fixed point is defined as a surface (or curve) such that if the initial condition is on the surface, the state asymptotically reaches the fixed point as $t \to \infty$, that is, when time is "run forward".

The unstable manifold of a fixed point is defined as a surface (or curve) such that if the initial condition is on the surface, the state asymptotically reaches the fixed point when time is "run backward" (as $t \to -\infty$).

2. Time running forward is following the arrows in the phase portrait.

Time running backward is following the reversed arrows, equivalent to hitting the REWIND button on your DVD player / VCR.
Case 8

\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \dot{y}_1 = y_1 y_1 \]
\[ \dot{y}_2 = 0 \]

Here, the fixed point is not unique.
Any point \((0, y_2)\) is a fixed point.

\[ y_1 \]
\[ y_2 \]

Line of fixed points.
SPECIAL BOUNDARY CASE

WHEN THERE ARE NOT ENOUGH EIGENVECTORS

\( y = Ay \), \( A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \)

Usually, we can find eigenvalues \( \lambda_1, \lambda_2 \) and distinct eigenvectors \( v_1 \) and \( v_2 \) associated with the eigenvalues.

But sometimes there is only one distinct eigenvector. An example follows.

\( A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \)

Eigenvalue
\( \lambda^2 - (a + 1) \lambda + a^2 = 0 \)
\( \lambda_1 = a \), \( \lambda_2 = a \)

Eigenvectors \( V \), \( AV = \lambda V \).

\( V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \)

\( \Rightarrow \begin{bmatrix} av_1 + v_2 \\ av_2 \end{bmatrix} = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow v_2 = 0 \) \( \) and \( v_1 \neq 0 \)

Eigenvector \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

This is the only eigenvector.

If we used our usual formula for the general solution for \( y = AV \), we'd get the following:

\( y(t) = c_1 V_1 e^{\lambda_1 t} \)

Because this solution has only one free parameter \( c_1 \), it cannot be a general solution.
How to find the general solution?

\[
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = ay_1 + y_2
\]

\[
\Rightarrow \begin{cases} \dot{y}_1 = y_2(0)e^{at} \\ \dot{y}_2 = ay_2 \end{cases}
\]

Integrating factor = \( e^{-at} \). Multiplying the above eqn by \( e^{-at} \):

\[
\begin{cases} \dot{y}_1 - ay_1 = y_2(0)e^{at} \\ \frac{d}{dt}(e^{-at}y_1) = e^{-at}y_2(0) \Rightarrow e^{-at}y_1 = y_2(0)t + c_1 \end{cases}
\]

\[
y_1(t) = e^{at}\left[y_2(0)t + c_1\right]
\]

Clearly \( c_1 = y_1(0) \). \( \Rightarrow \)

General Solution

\[
y_1(t) = e^{at}\left[y_2(0) + y_2(0)t\right]
y_2(t) = y_2(0)e^{at}
\]

\[
\text{Phase portrait } \frac{y_2(t)}{y_1(t)} = \frac{y_2(0)}{y_1(0) + y_2(0)t}
\]

Plot \( y_1 \) vs \( y_2 \) for different initial conditions to get the following.

When \( a < 0 \)

what does the phase portrait look like?

(FP called a \text{DEGENERATE NODE})

When \( a > 0 \)

(unstable version of the portrait to the right)
Example 3

\[ \dot{x} + cx + kx = 0 \quad c > 0 \text{ and } k > 0 \]

\[ y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \]

\[ \dot{y} = Ay \]

\[ A = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \]

Eigenvalues
\[ \lambda^2 + (c) \lambda + k = 0 \]
\[ \lambda^2 + c\lambda + k = 0 \]
\[ \lambda_{1,2} = -\frac{c}{2} \pm \frac{\sqrt{c^2 - 4k}}{2} \]

Say \( c^2 - 4k < 0 \) so that the eigenvalues are complex conjugates.

\[ \lambda_{1,2} = -\frac{c}{2} \pm \frac{i \sqrt{4k - c^2}}{2} \]

General solution
\[ x(t) = e^{-\frac{ct}{2}} \left[ b_1 \cos \omega t + b_2 \sin \omega t \right] \]

where \( \omega = \frac{\sqrt{4k - c^2}}{2} \).

and \( b_1, b_2 \) depend on initial conditions.

\[ c > 0 \quad c = 0 \quad c < 0 \]

\[ x(t) \]

Decaying oscillation

Periodic oscillation

Growing oscillation
CLASSIFICATION OF FIXED POINTS (again)

IN 2D

FP $[0]$ is called a CENTER.

FP $[0,0]$ is called an UNSTABLE FOCUS.

FP $[0]$ is a STABLE FOCUS.

SADDLE POINTS

UNSTABLE NODES

UNSTABLE SPIRALS

CENTERS

STABLE SPIRALS

STABLE NODES

STAR NODES & DEGENERATE NODES
Drawing the Phase Portrait for Nonlinear Systems

Essentially the general way to draw the phase portrait for a nonlinear system $\dot{y} = f(y)$ is numerical – that is, solve the ODE for different initial conditions and plot $y(t)$ vs. $y_1(t)$.

But one can sometimes deduce the qualitative structure of the phase portrait by finding all the fixed points of the nonlinear system, determine the eigenvalues (eigenvectors) corresponding to

let us consider an example: Simple pendulum with small $c$

Equation: $\ddot{\theta} + g \sin \theta + c \dot{\theta} = 0$

$y_1 = \theta$
$\dot{y}_1 = y_2$
$y_2 = \dot{\theta}$
$\dot{y}_2 = -k \sin y_1 - cy_2$

where $k = \frac{g}{L}$

Fixed Points

$y_2 = 0$

$\sin y_1 = 0 \Rightarrow y_1 = 0, \pi, \pm\pi$

Let us consider the 3 fixed points $y_1 = 0$, $y_1 = \pi$, and

$y_1 = -\pi$.

$y_1 = 0$ is the stable equilibrium.

$y_1 = \pm\pi$ are two different representation of the inverted unstable equilibrium.
LINEARIZATION AND LINEAR STABILITY

We have looked at this before:

At \((0,0)\), the Jacobian matrix is

\[
\begin{bmatrix}
0 & 1 \\
-k & -c \\
\end{bmatrix}
\]

We know that the eigenvalues are

\[
\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}
\]

For sufficiently small \(c\), these eigenvalues will be complex conjugates (damping) and the fixed point will be a **STABLE SPIRAL**.

At \((\pi,0)\), the Jacobian matrix is

\[
\begin{bmatrix}
0 & 1 \\
k & -c \\
\end{bmatrix}
\]

The eigenvalues are

\[
\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}
\]

\(\tau = \kappa\), \(\Delta = -k\), \(\tau^2 - 4\Delta > 0 \Rightarrow \text{SADDLE POINT. (UNSTABLE)}\)

To calculate eigenvectors let us make the parameters precise. For simplicity

\(c = 1\), \(k = 2\), \(\lambda_1 = -1 + \sqrt{9} = 1\), \(\lambda_2 = -1 - \sqrt{9} = -2\).

Eigenvectors are respectively

\(\begin{bmatrix}
\sqrt{2} \\
1
\end{bmatrix}\) and \(\begin{bmatrix}
-0.4472 \\
0.8944
\end{bmatrix}\)

\(\Rightarrow\) **STABLE DIRECTION**

\(\Rightarrow\) **UNSTABLE DIRECTION**
STABLE AND UNSTABLE
MANIFOLD FOR, SADDLE POINT.

PHASE PORTRAIT NEAR
$(\pm \pi, 0)$

PUTTING THE PICTURES TOGETHER.

So far so good. We can get a picture like above by following a
sequence of well-defined steps.

But to complete the picture, we have no general technique
that always works, except for computing the phase portrait
by solving the ODE (say numerically).

In this particular case, we can do a little better by appealing
to physical intuition and knowledge of the nature of the
solutions.

If you start from the inverted position $(\pm \pi, 0)$ and give it an
arbitrarily small perturbation, the pendulum oscillates and eventually
comes to rest in the stable bottom position $(0, 0)$.

In other words, the saddle points are connected to the
stable spiral.
Connecting the saddles to the spiral (two different spirals).

Adding the rest of the saddle.
The following are some of the most common bifurcations in 2D dynamics, very analogous to the corresponding 1D bifurcations.

In the following examples, you will notice that all the “bifurcation action” is happening along the x direction, whereas in the y direction, there is only simple stable dynamics.

Thus, it may superficially seem that these are ‘special examples.’

But they are not really that special. Rather, they are called ‘normal forms’ or ‘canonical forms.’ Any other dynamical system having a similar bifurcation can be transformed, at least locally, into these normal forms using a continuous (often differentiable) transformation of variables. That is, these simple examples (normal forms) have the same topological structure of every such bifurcation (locally).
Supercritical pitchfork

\[ \dot{x} = \mu x - x^3 \quad \dot{y} = -y \]

\( \mu < 0 \)
(one stable node)

As \( \mu \) is changed, one node becomes 3 fixed points, two nodes, and 1 saddle.

Transcritical bifurcation

Canonical example

\[ \dot{x} = \mu x - x^2 \quad \dot{y} = -y \]

\( \mu < 0 \)

Two fixed points
\( x^* = 0 \) - stable
\( x^* = \mu < 0 \) - unstable

\( \mu > 0 \)

Two fixed points
\( x^* = 0 \) - unstable
\( x^* = \mu > 0 \) - stable.

\( x^* = \) goes from stable to unstable as \( \mu \) is changed.

At \( \mu = 0 \) the phase portrait is topologically different from when \( \mu > 0 \) or \( \mu < 0 \).

Hence this is a bifurcation.

(even though the \( \mu > 0 \) and \( \mu < 0 \) portraits look similar)
Bifurcations in HW2 problem of stabilizing an inverted pendulum with a torsional spring

\[ \ddot{\theta} + b \dot{\theta} + g \sin \theta = 0 \]

Two fixed points (essentially)
\[ \theta^* = 0 \text{ and } \theta^* = \pi \]

Torsional spring with rest position \( \theta = \pi \) and stiffness \( k \).

\[ \ddot{\theta} + b \dot{\theta} + g \sin \theta + k(\theta - \pi) = 0 \]

(a) At \( \theta = \pi \), \( k > g/L \) for stability. \( g/L \) is the local "negative stiffness" due to gravity, so \( k \) for positive stiffness due to the spring needs to be high enough that \( k - g/L > 0 \).

For \( k > g/L \), \( \theta = \pi \) is the only fixed point (as can be seen from the first picture on the next page).

(b) What happens to the fixed points as the stiffness \( k \) is reduced?

Fixed point when \( g/L \sin \theta + k(\theta - \pi) = 0 \)

\( \sin \theta = \frac{gL(\pi - \theta)}{k} \)

\( \sin \theta = k(\pi - \theta) \cdot \sin \theta = \frac{gL(\pi - \theta)}{g} \)

Intersections of two curves: \((0, k4g)\) and \((\pi, 0)\)
Fixed points & stability as k is decreased.

Sufficiently shift limit \( \frac{KL}{g} > 1 \) if \( k > g/L \). Only 1 FP. at \( \theta = \pi \).

When \( \frac{KL}{g} > 1 \), the slope of \( \frac{KL}{g} (R-\theta) \) is greater than the slope of \( \sin \theta \).

Bifurcation Diagram so far.

\[ \theta^* = 0 \text{ in a stable FP for } \frac{KL}{g} \geq 1 \]

\( k \) just less than \( g/L \)

\( \frac{KL}{g} \) just less than 1

- There are 3 intersections \( \Rightarrow 3 \) FPs.
- One unstable
- and 2 stable.

(How to know that FP is unstable and FP2 and FP3 are stable? See end of this discussion on page 6.)
Difurcation diagram around $k = \frac{1}{3}$

As $k$ is reduced, other fixed points arise. But before that, let's see what FP2 and FP3 do as $k \to 0$.

We see that $FP_2 \to 0$ and $FP_3 \to 2\pi$.

Let us now consider how other fixed points arise as $k$ is reduced from $\frac{1}{3}$.

Critical slope at which FP4 - 7 are created.
The red line is when there are exactly 5 FPs.

for $k < \text{red line slope (blackline)} \Rightarrow 7 \text{ FPs}$

for $k > \text{red line slope (}) \Rightarrow 3 \text{ FPs}$

So just at the redline slope (e below it), 4 new FPs are created in 2 pairs.

- Call them $FP_4 - FP_7 \text{ as in previous page}$.

- $FP_4$ & $FP_6$ are roughly located as $+3.5\pi$
  when they are created.

- $FP_6$ & $FP_7$ are roughly located as $-1.5\pi$
  when created.

- When $k \geq 0$
  $FP_4 \rightarrow 3\pi$
  $FP_5 \rightarrow 4\pi$
  $FP_6 \rightarrow -\pi$
  $FP_7 \rightarrow -2\pi$

- $4\pi$ is $FP_5 \rightarrow \text{SADDLE-NODE BIFURCATION}$
  $3\pi$ is $FP_4 \rightarrow \text{SADDLE-NODE BIFURCATION}$

$-2\pi$ is $FP_6 \rightarrow \text{SADDLE-NODE BIFURCATION}$
$\text{(saddle or node appear out of thin air)}$
As $k$ is reduced even further, another critical slope (shown in red below) is crossed, which introduces 4 more fixed points, again in 2 pairs (one pair with $\theta > 0$ and another pair with $\theta < 0$).

FP8 and FP9 appear in a saddle-node bifurcation.

So do FP10 and FP11.

So extending the bifurcation diagram a bit:

Note that as $k \to 0$

- $FP_8 \to \pm \pi$
- $FP_9 \to 6\pi$
- $FP_{10} \to -3\pi$
- $FP_{11} \to -4\pi$
As \( k \) is decreased further, we get more and more fixed points, all arising in pairs of saddle-node bifurcations.

Ok, how did we know what the stability of the fixed points were?

**FACT** (you should be able to show this): The fixed points of

\[ x + bx + f(x) = 0 \]

and \[ bx + f(x) = 0 \]

are the same and **HAVE THE SAME STABILITY**.

Therefore, the fixed points and stability of

\[ \frac{\dot{x}}{1} \cdot \frac{\dot{y}}{1} + g \sin x + h(x) = 0 \]

are the same as the fixed points and stability of

\[ \frac{\dot{b} x + g \sin x + h(x)}{1} = 0 \]

So we can analyze the stability of the 1D equation:

\[ b \dot{\theta} = k(\pi - \theta) - g \sin \theta = h(\theta) \text{ say} \]

You can do a full 2D analysis to show that the fixed points are saddles, etc.